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MARSHALL'S *CETERIS PARIBUS* IN A DYNAMIC FRAMEWORK

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Abstract

The paper aims to propose a formalization of the concept of *ceteris paribus* (CP) by means of a dynamic model. The basic result of the analysis is that the CP clause may assume essentially different meanings according to (1) the kind of variables assumed to be "frozen" and (2) the length of the time horizon. It is then possible to distinguish, respectively, between an *historical* and an *endogenous* CP and, within the latter, between a *short-run* and a *long-run* CP. This double analytical distinction helps in understanding the role the CP clause plays in economic dynamics. Finally, the notion of *long-run* CP seems to suggest an extension of the standard view of the CP concept: interpreted as dynamics on manifolds, it still reduces the degree of complexity of a system even if variables "frozen" in it need not to be constant.

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KEYWORDS: *Ceteris Paribus,* Marshall, Dynamics, Partial Equilibrium.

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1 Introduction

Ceteris Paribus (CP) is a Latin expression formed by the words "Caeterus" (the other, the remaining, the rest) and "par" (similar, equal). It is commonly translated as "other things being equal", "other things being constant", "all else being equal", "other things the same", "in the absence of disturbing factors" and so on.

While there are still some doubts regarding who it was that first introduced this term in economics, historians of economics substantially agree that it was principally Alfred Marshall who popularized and spread the use of this clause in economics[1]. Explicit analyses and investigations of the CP-clause can be found in Book III and V of the Principles of Economics and in Marshall (1898), but references to and applications of the CP-clause are frequent in the whole body of Marshall's economic writings.

But what does "ceteris paribus" *really* mean when used in the context of economics? Although the CP-clause has always been, before and after Marshall, a necessary and indispensable analytical tool in economic theory, there is not much agreement on its "true" meaning[2]. Indeed, even after a careful reading of Marshall's texts, these doubts seem to be confirmed and a recurrent finding in any analysis of his work is that the CP-clause may assume similar but essentially different meanings. This paper is therefore an attempt to clarify the meanings and the relationship of the different usages of the CP-clause in Marshall. The basic result of our analysis is that it may assume different meanings according to (1) the kind of variables assumed to be "frozen" and (2) the length of the time horizon. We believe that this double distinction may help in understanding the role this clause plays within Marshall's dynamic vision of the economic system: not merely a simplifying analytical device but a fundamental aspect of his thought.

Three main tenets form the basis of our study.

First, the CP-clause in Marshall is not merely a simplifying analytical device but a fundamental aspect of his thought. When he invokes a CP-clause, Marshall does not mean to deny the existence of disturbing factors in a given situation of economic regularity. Instead, he aims at justifying the decision to ignore these disturbing factors because not considering them will

[1] See, for example, Whitaker (1987a), Persky (1990), Maki-Piimies (1998)

[2] "The term ceteris paribus has no clearly settled technical meaning among economists, so that any attempt to chronicle its usage would be both difficult and unrewarding" (Whitaker, 1987a, p. 396). See also Mäki e Piimies (1998) according to whom "ceteris paribus is a highly ambiguous expression, having a variety of usages" (Mäki e Piimies, 1998, p. 57) and Lipton who argues that "in fact there are many different types of cp laws and many different reasons for invoking them" (Lipton, 1999, p. 155).

provoke only a negligible error. Therefore, a "strong" interpretation of the CP-clause is not faithful to Marshall's epistemological view. The evaluation regarding how "negligible" an error is considered to be varies from context to context and crucially depends on the particular aims of each analysis. Unfortunately, or fortunately, no strict and a-priori methodological rule can be of any help in this evaluation.

Second, there is a clear nexus between the CP-clause and the problem of the interdependency of economic variables. The main reason behind why economists invoke the CP-clause lies in the fact that most economic phenomena are complex and if we want to describe and represent them as accurately as possible, we need to introduce many and reciprocally interdependent variables. On the other hand, it is often thought that isolating a certain set of variables with respect to the others - as CP clauses do - necessarily implies an interruption in the feedbacks among the several variables; this idea will be discussed and challenged here.

Third, since Marshall's view of economic science is crucially linked to the diffusion of economic changes throughout the community and since economic changes are not instantaneous but develop over a certain interval of time, the element of time is crucial in understanding the role of the CP-clause in Marshall's work. Starting from the supposition that the analysis of a marshallian economy cannot but be a dynamic one, this paper aims to propose a dynamic vision of the concept of ceteris paribus by developing its different meaning and usages throught articulating and classifying the idea of a dynamic flow.

From this point of view, this work differs from Friedman (1949) and Hausman (1989) where the Marshallian CP-clause, although not explicitly analysed, is interpreted by means of a static framework. Fisher and Ando (1962) and Ando and Simon (1962) propose a dynamic approach in the formalization of the issues involved by the CP-clause, but they are not related to Marshall's work. By contrast, this paper is linked to a recent branch of literature lead by Schlicht (1985, 1990), Dardi and Gay (1991) Gay (1991) and Cerina (2000, 2001) which deals with the Marshallian CP-clause by means of a dynamic system. This paper can be viewed as an extension and a deeper investigation of some of the ideas contained in these works.

The paper is divided in 8 sections. Following this introduction, the second section outlines the theoretical framework of the present analysis. The third section presents the first-level distinction within the concept of CP, that is to say the one between an *historical* and an *endogenous* CP-clause. We extensively discuss the former notion in the fourth section. The fifth section outlines the second-level distinction between a Short-run Endogenous CP-clause and a Long-run Endogenous CP-clause. The sixth and the seventh sections formalizes these two concepts and the eighth section concludes.

2 The theoretical framework

The economy is described in each date by a vector $s = (s_1, s_2, ..., s_k) \in S$ where S is an open subset of $\Re^k$ representing the *space of conceivable states of the economy.* Each state is described by k components.

We define a *vector field* $\dot{s}$ in S by associating to each state s the speed according to which each state is changing. We thus obtain a dynamic system in k differential equations:

$$\dot{s}_h = z_h(s_1, s_2, ..., s_k) \tag{1}$$

$$h \in K = [1, 2, ...k],\ z_h : S \to \Re \text{ e } S \subset \Re^k$$

which, in compact form, can be also written as

$$\dot{s} = z(s) \tag{2}$$

We assume that $z \in C^1$.

Once the vector field is defined, we can obtain the dynamic behavior of each state of the economy by deriving the *flow function* $\varphi : \Re^{k+2} \to \Re^k$ such that if $s = \varphi(t, t^\circ, s^\circ)$ then $\dot{s} = D_t\varphi(t, t^\circ, s^\circ) = z(\varphi(t))$ holds for each $t \in \Re$. The function $\varphi(t, s^\circ, t^\circ)$ tells us which is, at time t, the state of the economy whenever, at time t°, its state is s°. As a first approximation we assume that the system is autonomous so that it is possible to consider the flow as a function of the time and the initial state only and not even of the initial time: $\varphi(t, s^\circ)$.

$D_l z_h(s) = \dfrac{\partial z_h}{\partial s_l}$ indicates how $\dot{s}_h$changes as s_l is moving and it represents the element on the h^{th} row and l^{th} column of the jacobian matrix of the function $z : S \to \Re^k$. This matrix gives us an intuition the interdependence links among the several variables.

A *stationary state* is defined as the state in which *all the variables* remain constant, that is, a state $s^\circ = (\overset{\circ}{s}_1, \overset{\circ}{s}_2, ..., \overset{\circ}{s}_k,)$ such that, for each $h \in K$

$$\dot{s}_h = z_h(\overset{\circ}{s}_1, \overset{\circ}{s}_2, ..., \overset{\circ}{s}_h, ..., \overset{\circ}{s}_k,) = 0 \tag{3}$$

A *partial (and temporary) equilibrium state* is considered as a steady state localised in an area of the space state. In this state a subset of the total k variables remains constant while its complement modifies its state. If

p $(p < k)$ is the number of the variables assuming zero speed, such a state could be defined as a vector $\overset{\circ\circ}{s}$ such that:

$$\begin{aligned} \dot{s}_l &= z_l(\overset{\circ\circ}{s}_1, \overset{\circ\circ}{s}_2, ..., \overset{\circ\circ}{s}_k) \neq 0 \\ \dot{s}_h &= z_h(\overset{\circ\circ}{s}_1, \overset{\circ\circ}{s}_2, ..., \overset{\circ\circ}{s}_k) = 0 \end{aligned} \tag{4}$$

$$\text{for } l \in \{1, 2, ..., p\}, \;\; p < k \;\; \text{ and } \;\; h \in \{p+1, p+2, ..., k\}.$$

Each partial equilibrium state is temporary because each s_h is a function of the entire vector s and so the behavior of the h^{th} variable is not independent from the whole description of the system at the current state. The latter is in fact a result of the dynamic behavior, so far, of *all the other variables of the system.* Since $\dot{s}_l = z_l(s) \neq 0$, this movement will cause (with probability 1) a change in the state of all the other variables and so also in the variable s_h.

In order to guarantee the highest level of generality, we assume that for each pair (h, l) $h \neq l$, $D_l z_h(s)$ is not constantly zero at all points of S.

3 A first-level distinction: Exogenous and Endogenous CP

Marshall's basic idea that "great part of economic science is occupied with the diffusion throughout the community of economic changes which primarily affect some particular branch of production or consumption"[3], serves as our starting point. He develops this very idea in several pages of the *Principles* by means of some conceptual experiments, the most famous of which is probably the one concerning the fish industry[4]. By following the steps of his reasoning, different usages of the CP-clause emerge naturally.

Marshall's approach starts from a stationary economy. The mathematical definition of a stationary state we have given above does not allow us to capture the wealth of meanings that this analytical tool has in Marshall's thought. A stationary state is viewed by Marshall as that particular state of the world in which all the variables that describe the dynamic behaviour of the economic system are at their "normal" equilibrium level. That is, at "the average value which economic forces would bring about if the general conditions of life were stationary for a run of time long enough to enable them to work out their full effect"[5]. In other words, it is the state which the

[3] Marshall (1961, Vol.I, p.413).
[4] *Ibid.* p.369.
[5] *Ibid.* p.347.

economy naturally tends to if no unexpected event happens in the meanwhile. Once the economy reaches a stationary state, there is no reason for it to leave this state as long as agents' expectations are continuously confirmed and hence, again, no unpredicted event occurs.

But agents cannot foresee everything and thus, sooner or later, the very passage of time gives rise to an unexpected and exogenous shock (for example a disease affecting the large part of the bovine stock) which affects at first just a part of the economic system (in our example fish and meat production and fish and meat consumption).

Economic agents operating in the branches most directly susceptible to the effects of this shock are those that will be primarily affected and, as soon as they realize that "things are changed" and that new arbitrage operations are now feasible, they seek to exploit this situation to their own advantage. In doing so, they modify the "normal" equilibrium value of the variables they can directly affect (in this case: fish and meat production and consumption and fish and meat equilibrium prices) and thus the actual values of these variables move towards their *new* "normal" equilibrium values[6].

However, markets are highly interdependent and therefore an exogenous shock - although localized in a subset of the economic system - at the highest level of generality will actually provoke a change in the equilibrium value of the whole set of variables. Therefore the movement will be immediately spread over the whole economic system, albeit at different speed and with different intensity according to the logical proximity of the shock.

Let's now go back to the previous formal definition of steady state: being interested in formalizing Marshall's approach to economic dynamics, we must realize we have a problem with (3). The state s° is in fact an *invariant*, that is, there is no active force capable of perturbating the economy and creating movement within the system. Since everything is fixed in this state, no interdependence feedback can be activated and, therefore, there is apparently no need to invoke any CP-clause.

In order to "create" movement in the system, we need to modify the (1) introducing an element of historical change in the form of a random shock which affects the vector field. As an unexpected event occurs, the agents' stock of information changes and a new set of expectations is ought to be elaborated. This change in the information set disturbs the peaceful existence of agents and gives them new chances to exploit economic conveniences.

Formally, the *autonomous* system expressed in (2) will be transformed

[6] Marshall adopts here an implicit assumption according to which, for each possible vector field, there is a unique and stable steady state. We adopt the same assumption in the rest of the paper without discussing how restrictive it may be.

into a *non-autonomous* one.

$$\begin{aligned} \dot{s} &= z(s, \xi(t)) \\ s &\in S,\ \xi(t) \in \Theta, \dot{s} \in \Re^k \times \Theta \end{aligned} \tag{5}$$

$\xi(t)$ is a stochastic process[7] describing the *state of the available information* at time t, a function that tells us "what is known and true" in correspondence with any date t. Since this function could be thought of as having either quantitative or qualitative values, its range will not necessarily be a subset of $\Re$ or of any of its cartesian products. As a novelty occurs, the state of the information will change and this change will shift the vector field of the economy modifying the equilibrium values of the variables[8]. With this assumption, the initial steady state is no longer invariant.

We are now able to single out two substantially different ways in which the CP-clause can be invoked. We first notice that, once the whole set of economic variables is in motion, "the forces to be dealt with (...) are however so numerous, that it is best to take a few at time; and to work out a number of partial solutions as auxiliaries to our main study"[9]. This is why economists should "isolate" the dynamics of some endogenous variables with respect to some other and, "for a time", concentrate only on the dynamics of the former. In doing so, they actually neglect some of the inter-linkages between different sets of endogenous variables and (implicitly or explicitly) assume that a given set of variables, those "frozen" in the CP-clause, remains in its equilibrium even after the shock. Since the aim of this clause is essentially to temporarily restrict the analysis to a subset of the endogenous variables, we call this clause **endogenous** CP.

But Marshall's words, and his approach to economic dynamics, suggest another meaning for the CP-clause: once the possibility of exogenous shocks has been introduced, we cannot a-priori rule out the possibility that, during the adjustment process triggered by the first shock, other shocks will occur, hence providing overlapping effects on the dynamics of the variables.

Since it is very difficult to distinguish which effects are endogenously determined and which are instead to be ascribed to a further exogenous

[7] It is plausible to assume that these "historical" changes do not occur continuously through time but only in isolated points of the real line so that $\xi(t)$ can be thought of as a *Wiener* process.

[8] A more complex modelling strategy is to assume that the vector field can be modified also by a change in the function $\Psi(t^*, \xi(t^{**}))$ representing the "elaboration" by the agents at time t^* of the information $\xi(t^{**})$ available at time t^{**}. In order to avoid any further complication and to focus on the definition of *ceteris paribus*, we assume that the elaboration of a novelty is instantaneous and univocal. That is $\Psi(t^*, \xi(t^{**})) = \xi(t^{**})$ and $t^* = t^{**}$.

[9] Marshall (1961, Vol.I, p.xiv).

shock, the economist should initially limit himself to the analysis of the pure adjustment process triggered by a single shock. This actually forces us to invoke a second kind of CP-clause, which we call **exogenous** CP, according to which no other exogenous shocks will occur "in the interval between an economic change and the full development of its effects”[10]. By this device, the shock which triggers the adjustment process under observation is actually assumed to be the *first and last shock in history*[11]. We now focus on these two different notions starting from the latter.

4 Exogenous CP

Going back to our theoretical framework, the exogenous CP-clause formally requires that the vector field does not change during the adjustment process of the variables following a previous random shock and, therefore, the steady state values of the k variables remain constant. Let’s consider the following Cauchy problem

$$\begin{aligned} &\dot{s} = z(s, \xi(t)) \\ &s(0) = s^{\circ} \end{aligned} \tag{6}$$

and assume that there is a shock in $t = 0$, the *first* and *last* of the history. Therefore $\xi(t)$ changes, say, from ξ° to ξ^{1}

$$\xi(t) = \begin{cases} \xi^{\circ} \text{ for } t < 0 \\ \xi^{1} \text{ for } t \geq 0 \end{cases} \tag{7}$$

We assume $\dot{s}(0) = 0$ and since $\partial\xi^{\circ} = \xi^{1} - \xi^{\circ}$ is *the first* shock in history, $\xi(t)$ must assume, for $t < 0$, a value which guarantees the maintainence of the steady state s°

$$\xi^{\circ} : \varphi(t, s(0); \xi^{\circ}) = s^{\circ} \text{ for every } t < 0 \tag{8}$$

and, therefore, $\dot{s} = z(s^{\circ}, \xi^{\circ}) = 0$ for every $t < 0$.

The change $\partial\xi^{\circ}$ initially provides a shift in the vector field and hence a change in the *equilibrium* values of the variables, but *not* in the *actual* values of the variables. The state of the system in $t = 0$ is therefore still s°, but this is no longer an equilibrium value for $z(s, \xi^{1})$.

$$\begin{aligned} &\varphi(0, s(0); \xi^{1}) = s^{\circ} \\ &\dot{s} = z(s(0); \xi^{1}) \neq 0 \end{aligned} \tag{9}$$

[10] Marshall (1961, vol.II, p.362).
[11] Dardi (2003).

But since $\partial\xi^{\circ}$is also *the last* historical change, we must have

$$\varphi(t, s(0); \xi(t)) = \varphi(t, s(0); \xi^1) \text{ for every } t \geq 0 \tag{10}$$

Therefore, provided that there will be no other shocks in the information set, the effect of this change in the solution of the Cauchy problem will be the following[12]

$$\partial\varphi = \varphi(t, s(0); \xi^1) - s^{\circ} \tag{14}$$

We can therefore state the following

Definition 1 *(Exogenous CP) An Exogenous CP assumption is satisfied if (8) and (10) holds.*

Notice that, if satisfied, this assumption allows us to consider the non-autonomous system (5) as if it was autonomous. In fact, since we will analyse the dynamic of the system only from $t = 0$ on, the date in which there is the first and last historical shock, we are allowed to express the solution as a function of time and the initial state only and not even of the state of information. That is

$$\varphi(t, s(0); \xi(t)) = \varphi(t, s(0); \xi^1) = \varphi^1(t, s^{\circ}) \text{ for every } t \geq 0 \tag{15}$$

For the same reason, the vector field can be expressed as a function of the state only and not even of time

$$z(s, \xi(t) = z(s, \xi^1) = z^1(s) \text{ per ogni } t \geq 0 \tag{16}$$

[12] We can't rule out the possibility that $\varphi(t, s(0), \xi^1)$ assumes, for a given t, the value $s(0)$. However, this value can't be a steady state for $z(s, \xi^1)$: if for any t' $\varphi(t', s(0), \xi^1) = s(t') = s(0)$, then $z(s(0), \xi^1) \neq 0$.

For the very same reason, the new steady state value will be different from $s(0)$.

If s^1 is the unique limit point of the solution $\varphi(t, s(0), \xi^1)$, the new steady state value, assuming uniqueness and stability, will be

$$\lim_{t \to +\infty} \varphi(t, s(0), \xi^1) = s^1 \tag{11}$$

Since it must be $z(s^1, \xi^1) = 0$, from (3) we have that

$$s^1 \neq s(0) \tag{12}$$

The final effect of the novelty $\partial\xi^{\circ}$ is given by the comparison between the two steady state values:

$$\lim_{t \to +\infty} \left(\varphi(t, s(0), \xi^1) - \varphi(t, s(0), \xi^{\circ})\right) = s^1 - s(0) \tag{13}$$

and this difference cannot be zero.

Hence, by using the Exogenous CP-clause assumption, we are able to go back to an autonomous system.

Marshall is of course aware that this is a very strong assumption, especially when the process under examination spans a long period of time since "in the long period (...) the general conditions of equilibrium are likely to be modified by external changes"[13]. Nonetheless, the assumption is somehow unavoidable since, when studying the effects of a disease affecting the bovine stock on the price of fish, we are not only obliged to isolate the effect of this shock from the effect on the same variable of, say, a terrorist attack, but we are also interested in doing this.

So as, with reference to the study of the oscillatory movements of a tower, it would be meaningless to explicitly introduce a CP-clause on earthquakes[14], similarly there seems little point in expressing, with reference to the dynamic analysis of a certain set of economic variables, a CP-clause on a sudden strike, on an unexpected tax, on a war, on a cattle plague, on a falling of meteoric stones or on anything which seems to be unpredictable and independent on the dynamics of the state variables. As already mentioned, it seems somehow obvious that these kind of dynamic studies cannot take into account the outcomes of events which are, by definition, completely unpredictable. Basically, there is always an historical CP-clause at work to remind us that no model can be completely exhaustive since there is always something outside the model which can affect its dynamic behaviour but can neither be predicted nor even conceptualised.

Indeed, even if in real historical situations shocks tend to arise in relatively quick succession and hence we never observe a complete adjustment triggered by a single shock, the exogenous CP-clause represents a compulsory starting point in order to study the whole process of historical change which is nothing but the overlapping of pure adjustment dynamics. In other words, the exogenous CP is a necessary condition for the study of a pure adjustment process.

Moreover, even though the exogenous CP is not so useful in the long run, it is also true that the shorter the time horizon, the more acceptable the exogenous CP is. In other words, as Marshall suggests, "the changes in the general economic conditions around us are quick: but they are not quick enough to affect perceptibly the short-period normal level about which the price fluctuates from day to day: and they may be neglected [impounded in ceteris paribus] during a study of such fluctuations"[15]

[13] Marshall (1961, vol.II, p.524).
[14] Whitaker (1987a, p. 396).
[15] Marshall (1961, vol.I, p.369).

The latter is a typical example where the element of time and the time-period framework is used by Marshall in order to reduce the complexity of the analysis without compromising its quality. By using the exogenous CP-clause, Marshall does not mean to deny the existence of disturbing effects. He rather aims to identify the conditions under which we might successfully behave as if these disturbing factors were not there. Marshall replicates, with different arguments, this epistemological position in its analysis of a pure adjustment process. It is here that time-period analysis is literally invented by Marshall in order to justify the soundness of the endogenous CP-clause.

5 A second-level distinction: Short-run and Long-run Endogenous CP

From this section on we will focus on the logical structure of partial equilibrium analysis with respect to a pure adjustment process towards new equilibrium values after a single and unpredicted shock has occurred at time $t = 0$. Therefore, a necessary condition in order for this analysis to be feasible is for the exogenous CP to be introduced.

In the context of a pure adjustment process, the main feature of time-period analysis is well represented by Marshall's idea that although "the changes in the volume of production, in its method, and in its cost are ever mutually modifying one another", nonetheless we can control and single out the action of the different forces because "all these mutual influences take time to work themselves out and, as a rule, no two influences move at equal pace"[16].

In other words, Marshall believes that variables, given their intrinsically different speed, can be classified and grouped with reference to the time (short, medium or long) which they require to fully develop the effect (that is, to reach their new equilibrium value) induced on them by an exogenous shock. Once they are classified this way, we can "impound in the ceteris paribus those forces which are of minor importance relatively to the particular time we have in view"[17].

One of the major implications for this view is that, for relatively short periods of time after an exogenous shock (in the short run), we are allowed to neglect the feedback effects that the dynamics of the relatively faster variables (for instance prices, unskilled labour supply and demand) exert on the dynamics of the relatively slower ones (for instance quantities, physical

[16] Marshall (1961, vol.I, p.368).
[17] Marshall (1898, p.47).

and human capital stock). We can then behave as if the slow variables were still in their "old" equilibrium while the fast ones move towards their "new" equilibrium. Hence, for sufficiently short periods, it is generally not only convenient but also plausible to treat relatively slow adjustment processes as if they were given and therefore freeze them in the CP "pound". This is, roughly speaking, what we mean by *Short-run Endogenous CP.*

But what happens when time horizons get wider? Reasonably, we might be progressively compelled to release more and more variables from the CP-clause, "for even indirect influences may produce great effects in the course of a generation"[18]. Thus, sooner or later, there comes a time in which the dynamics of the slow variables become relevant enough to substantially affect the dynamics of the fast variables. We are not even allowed to behave as if, in the long run, the fast variables remained constant at their new equilibrium value since it would imply that the dynamics of the slow variables exert no feedback on the dynamics of the fast ones, which is far too strong an assumption. Hence, apparently, there seems to be no room for any CP-clause in the long run.

However, there are many passages in which Marshall suggests neglecting those variables whose movement is too fast with reference to the length of the period we have in mind and to "sacrifice some precision of detail for the sake of being able to take at one glance a broad and comprehensive survey of the ultimate tendencies under discussion"[19]. Describing the famous example of the long run effect of a cattle plague on the fish industry, he suggests that is necessary to "concentrate our chief attention on causes which act slowly but continuously" and to "put aside fluctuations that come and go in a year or two". In order to justify this restriction in the number of variables to be taken into account, Marshall invokes an effective mechanical analogy:

> Having then got a compact and definite problem of equilibrium about a centre which does indeed move slowly, but the movements of which we have for the time neglected, we next take account of those movements; and thus gradually get a broader view of oscillations about a centre which is itself moving and perhaps oscillating in a longer period of time about another centre: somewhat as the moon moves round the earth, which itself in a longer period moves round the sun. But the sun itself is not fixed. It is moving and perhaps with an oscillatory movement about some very distant centre. And so while market prices oscillate about a position of market equilibrium, which perhaps oscillates about a position of short period equilibrium, that position in its turn may not remain stationary, but may move onwards in one direction, or may oscillate more slowly round a position of long-period normal equilibrium; and that again in its turn may itself be liable

[18] Marshall (1898, p.47).
[19] Marshall (1961, vol. II, p. 394).

to slow changes, possibly having an oscillatory movement, the period of which ranges over many generations or even centuries (Marshall, 1961, vol. II, p. 394-395)

These words seem to imply that, according to his view, long-period analysis (the movement of the sun) should not really be interested in the actual dynamics of the fast variables (the movement of the earth), since these variable are so fast that, when the time horizon is long enough, their effective value can be well approximated by their equilibrium value. And since this equilibrium is moving, long-period analysis should refer directly to the dynamics of the equilibrium values of the fast variables assuming the latter to be actually in equilibrium.

Thanks to this analytical device, which we define as *Long-run Endogenous CP,* Marshall is actually able - still in the long run - to approximate the behaviour of the whole economic system by focusing on one if its parts. This happens because the partial equilibrium values of the fast variables, in the proximity of which they are assumed to get trapped, are indeed a function of the particular values taken by the slow variables; hence, they can be deduced by the actual values assumed by the slow variables, which need not be in equilibrium.

Let's now formalize these intuitions by focusing first on the Short-run.

6 Short-Run Endogenous CP

6.1 Fast and slow variables

Assume a shock at time $t = 0$, the first and last in the history. This shift does not alter the values of variables but modifies the equilibrium values of the system by setting the velocities of the variables to a non zero value. Since the velocity of each of the variables depends on the state of *all* the variables (and so also of s_1), even if $z_1(s(0), \xi^1) \neq 0$ but $z_h(s(0), \xi^1) = 0$ for every $h \in \{2, ..., k\}$, the temporary equilibrium state assumed in $t = 0$ by these $(k - 1)$ variables would not *in general* be preserved as s_1 moves and, as a consequence, every component assumes nonzero velocity at the instant immediately following $t = 0$.

Since the number of components K involved in the description of the system can be extremely high, this situation could be very hard to deal with[20]. For this reason, when studying the evolution of a certain system over time, usually economists concentrate on the dynamics towards equilibrium of a certain set of variables, called *state variables*, given some *fixed values* of

[20]Marshall (1961, vol.I, p. xiv).

a certain set of *parameters* that describes the "environment" in which the system is embedded. Obviously, the nature of these parameters changes with respect to the aims and the context of the analysis but usually in economics they are introduced in order to describe preferences, technology, institutional settings (as in long run economic growth models) and, if the time horizon is sufficiently short (as in demand driven short run models), also capital stock (physical and human) and prices.

We can formalize this idea in the following way. We partition the $k-$ dimensional vector s obtaining two vectors, $x = (x_1, x_2, ..., x_n) \in X \subset \Re^n$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \in \Gamma \subset \Re^m$. We can think of the first vector as representing the new *state variables* of the system, namely the set of variables whose dynamics is considered essential for the description of the evolution of a certain system over time. The second vector can instead be thought of as a vector of parameters, describing the environment in which the system is embedded. We can order the variables in such way that

$$\begin{aligned} s &= (s_1, s_2, .., s_{n,} s_{n+1}, s_{n+2}, ..., s_{n+p-1,} s_k) = \\ &= (x; \alpha) = (x_1, x_2, ..., x_n; \alpha_1, \alpha_2, ..., \alpha_m) \in X \times \Gamma \\ \text{with } X \times \Gamma &= S \subset \Re^n \times \Re^m = \Re^k \end{aligned}$$

Without loss of generality, we can therefore represent our dynamic system in the following way

$$\begin{cases} \dot{x} = f(x; \alpha) \\ \dot{\alpha} = g(x; \alpha) \end{cases} \tag{17}$$

Notice that nothing has changed in the dynamic behaviour of our economy, since the (17) is just another way to describe the same system. But this formulation suggest that the parameters themselves have a *dynamics on their own* which can be affected by the dynamics of the state variables. However, Marshall suggests that only the transitional dynamics of the state variables is needed to be studied, abstracting from the dynamics of the parameters which are considered to remain "almost" constant.

Immediately after the first and last shock in history at time 0 which moves the field vector of the state variables only, the economy will be in a state $s^{\circ\circ} = (x^{\circ\circ}, \alpha^{\circ\circ})$ such that

$$\begin{aligned} \dot{x} &= f(x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}; \alpha_1^{\circ\circ}, \alpha_2^{\circ\circ}, ..., \alpha_m^{\circ\circ}) \neq 0 \\ \dot{\alpha} &= g(x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}; \alpha_1^{\circ\circ}, \alpha_2^{\circ\circ}, ..., \alpha_m^{\circ\circ}) = 0 \end{aligned} \tag{18}$$

Then, the endogenous CP-clause we have mentioned above can be formally expressed by requiring that this situation of partial equilibrium is mantained during the transitional path of the state variables x towards their new equilibrium values and. This is tantamount to saying that the set $G = \{(x;\alpha) : g(x;\alpha) = 0\}$, identifying the subset of the state space such that the parameters have zero velocity, is *positively invariant*[21].

The condition according to which the set G is positively invariant is extremely restrictive. In particular, in the most general case in which $m \geq n$, this is true only if, trivially,[22]

$$\frac{\partial g_j}{\partial x_i} = 0 \tag{19}$$

for every $i \in N, j \in M$ and for every $x \in X$

The (19) requires the independence of the velocity of the parameters from the state variables and implies therefore that the jacobian matrix of the system is diagonal. If this is the case we can represent the system by modifying the vector field of the slow variables:

$$\begin{aligned} \dot{x} &= f(x;\alpha) \\ \dot{\alpha} &= \overset{\circ}{g}(\alpha) \end{aligned} \tag{20}$$

where now $f : X \times \Gamma \rightarrow \Re^n$ e $\overset{\circ}{g} : \Gamma \rightarrow \Re^m$.

The assumption according to which the jacobian matrix of the system is diagonal seems to solve the problem of the CP by simply denying it and, moreover, is too restrictive. It's not difficult to find some important actual situation in which the dynamics of the parameters can be reasonably considered endogenous[23]. This is even more the case when the time horizon is long enough.

However, a weaker version of this assumption may be accepted when the time horizon is short enough to guarantee that a subset of components of s, the parameters vector α, does not change its state in a "relevant" way. The underlying intuition, already mentioned above, is that although there is a complete interaction among the k components of s, not all these components move at the same speed. There are some components, the parameters, which

[21] Formally a set G is a positively invariant if, for each (x, α) in G, the flow $\varphi(t, x, \alpha)$ is defined and remains in G for every $t \geq 0$.

[22] Since every explicit ordering relation between the number of parameters (m) and the state variables (n) would be hard to interpret economically, it seems reasonable to take this condition into account. A less restrictive condition, holding only for $m < n$, is formulated in the appendix.

[23] The most typical and devastating example supporting this argument is the Lucas critique (Lucas, 1976).

can be reasonably thought to move slower than others, and this intrinsic difference can be justified by the existence of some technological, psychological, institutional or historical rigidities[24]. If we assume that the difference in velocity among these two (or more) groups of components of s is sufficiently consistent, we can group them according to the time period (short or long) they require to fulfil their effect. In other words, the element of time mitigates the interactions among the several components of s. Therefore, since the slow variables (the parameters) cannot change their state in a relevant way, for sufficiently short periods and for given values of the partial derivatives of the system, the feedback effects on the fast variables are not "significantly" activated.

6.2 Effective and Modified flow

The idea according to which in the short run the "slow variables" or parameters α modify their state in a non-relevant and therefore negligible way, seems to invoke the notion of *threshold.*

Again, let's assume that at time zero there is the first and last shock in the economic system's history. We focus on the x vector and our aim is to find a condition such that the difference between the *effective* dynamics of x (with no constraints on the dynamics of α) and the *modified* dynamics of x (with each α_j fixed at its initial steady state value $\overset{\circ}{\alpha_j}$) is "negligible". Such a condition would not deny the movement of the parameters (slow variables) but it requires that their effect on the dynamics of the state variables is so small that "we don't lose too much" in considering the parameters to be fixed, at least for a short interval of time.

The effective dynamics of x is the result of the following Cauchy problem

$$\begin{cases} \left(\dot{x}, \dot{\alpha}\right) = [f(x,\alpha), g(x,\alpha)] \\ (x(0), \alpha(0)) = \left(\overset{\circ}{x}, \overset{\circ}{\alpha}\right) \end{cases}$$

The solution will be any function $\varphi(t, \overset{\circ}{x}, \overset{\circ}{\alpha}) = (x(t), \alpha(t)) \in \Re^{n+m}$ which in the state (x, α) at time t assumes the velocities required by the vector field (f, g). The effective dynamics of x can therefore be represented by the first n value of the previous flow $\varphi(t, \overset{\circ}{x}, \overset{\circ}{\alpha})$. The dynamics obtained without posing any further constraint and without making any assumption on the dynamics of α are referred to as $x(t)$..

The modified dynamics of x is the one obtained in the particular case where each α_j remains fixed at its initial steady state value. In other words,

[24]Marshall (1961 vol.I, p. 368, V, v, 3).

we aim at studying the dynamics of the system "as if" the set G is positively invariant and therefore the movement of x does not affect the velocity of α. The modified dynamics of x is the result of the following Cauchy problem

$$\begin{cases} \left(\dot{x}, \dot{\alpha}\right) = [f(x, \alpha), \tilde{g}(x, \alpha)] = [f(x, \alpha), 0] \\ (x(0), \alpha(0)) = \left(x^{\circ}, \alpha^{\circ}\right) \end{cases}$$

The solution will be any function $\widetilde{\varphi}(t, x^{\circ}, \alpha^{\circ}) = (\widetilde{x}(t), \alpha^{\circ}) \in \Re^{n+m}$ assuming the velocities required by the vector field $(f, \widetilde{g})$ in point (x, α) at time t Again, we are interested in the first n values of this function since they represent the trajectory of x in the particular case where each α_j is fixed, for $t \geq 0$, at the value α_j°. The last m values of $\widetilde{\varphi}(t, x^{\circ}, \alpha^{\circ})$ are in fact the (constant) values of the vector α° since, by assumption, $\widetilde{\alpha}(t) = \alpha^{\circ}$ $\forall t \geq 0$.

Notice that the field vector of x, *per se*, has not been modified since velocities associated to each state of $X \times \Gamma$ remain the same. However, we now restrict our attention to the subset $X \times \left\{\alpha^{\circ}\right\}$ of the space state. We define the *modified dynamics* of x as $\tilde{x}(t)$.

6.3 The negligibility condition

We can now think of our short-run endogenous Cp-clause as the requirement according to which, for a certain interval of time, the effective dynamics of x does not diverge to "any substantial degree" from the modified dynamics of x. Only if this condition holds, in fact, is it reasonable and useful to concentrate, for short periods of time, on the modified dynamics of x, which is a simplified version of the actual one[25].

Let's introduce a *distance function* between the vector $x(t)$ and the vector $\tilde{x}(t)$. We are allowed to do this since we are dealing exclusively with metric spaces.

$$d(t, x^{\circ}, \alpha^{\circ}) = \|x(t) - \tilde{x}(t)\| = \sqrt{\sum_{i=1}^{n} [x_i(t) - \tilde{x}_i(t)]^2} \tag{21a}$$

where $d : \Re^{+} \times [X \times Y] \rightarrow \Re^{+}$.

The function d has some important properties. First of all it is a *continuous* function since its the norm of the difference of two flows which are C^1 in t and in the initial states. However, being a norm, d is not differentiable when $x(t) = \tilde{x}(t)$, namely, when it takes zero values.

Moreover, $d(0, x^{\circ}, \alpha^{\circ}) = 0$ since the initial states of the two dynamics coincides. Nevertheless, d may also assume zero values for some $t > 0$ since

[25] See also Schlicht (1985, p. 46).

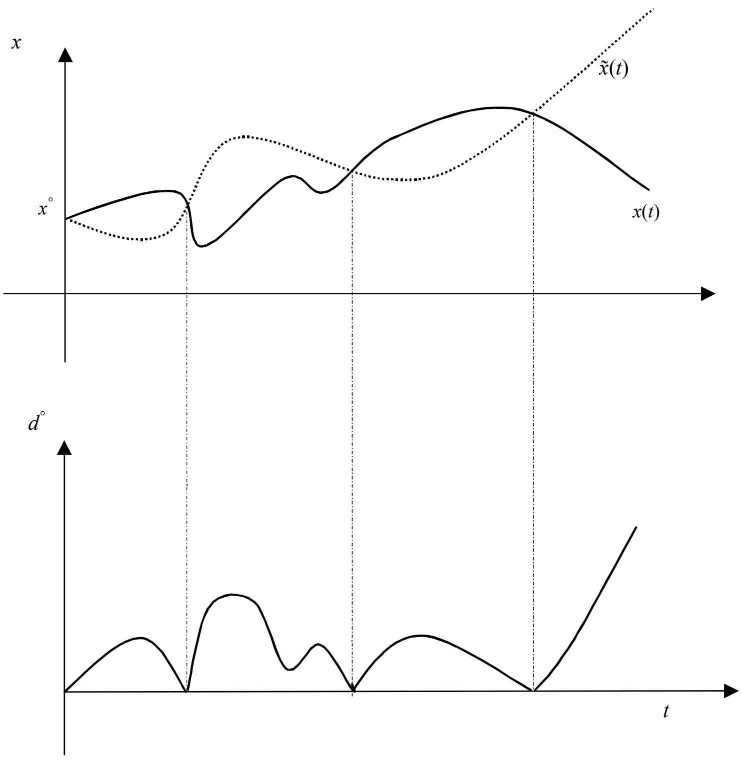

Figure 1: The distance function for $n = 1$.

we cannot exclude a-priori that the two flows, after having assumed different values for a certain time interval, will share the same state at some time t'. We can represent this idea graphically for $n = 1$.

Finally, we know that $\lim_{t \to +\infty} d(0, x^{\circ}, \alpha^{\circ}) \neq 0$ because $x(t)$ and $\tilde{x}(t)$ have different steady state values. Nothing more can be said for any $t \in (0, \infty)$.

The value of d depends, at a given point in time, on the particular choice of the vector of initial states. However, as far as our aims are concerned, we are not interested in the behaviour of the distance function as the initial states change but rather in the evolution over time of d with fixed initial states. Since the vector of initial states is the same for both the dynamics, we can somehow choose it arbitrarily. Hence, with fixed initial states

$$d(t, x(0), \alpha(0)) = d(t, x^{\circ}, \alpha^{\circ}) = d^{\circ}(t) \tag{22}$$

with $d^{\circ} : \Re^{+} \to \Re^{+}$

The negligibility condition, representing our *short-run endogenous CP-clause*, can be formalized by requiring that the function $d^{\circ}(t)$ is upper bounded by a certain tolerance threshold δ. This threshold must of course be carefully chosen taking into account the nature of the economic variables involved in the analysis and the context to which it refers since its important role is to divide between the case in which the dynamics of the state variables is notably affected by the dynamics of the parameters and the case in which it is not.

Definition 2 *(Short-run Endogenous CP) For a given threshold $\delta \in \Re^+$, the short run endogenous CP-clause is verified within an interval $[0, t')$ if*

$$\sup_{t \in [0,t')} \left[d^{\circ}(t) \right] < \delta \tag{23}$$

If this condition holds, then we are allowed to use the simplified and modified dynamics $\tilde{x}(t)$ instead of the effective, but more complex, $x(t)$.

Needless to say, this approximated interpretation of the short-run endogenous CP-clause cannot be considered to hold for every $t \geq 0$. For a proper choice of δ, it is reasonable to believe that there is a certain t'' such that the previous condition does not hold any longer. We are particularly interested in the initial maximum interval in which the Short-Run Endogenous CP-clause holds. A reasonable assumption is that, once the threshold has been overcome for the first time, the dynamics of x is irreparably and irreversibly affected by the dynamics of α and therefore, from this date on, we cannot trust $\tilde{x}(t)$ any longer.

Using the theorem of continuity of the flow, it is possible to show[26] that such a maximum interval always exists. Let's call $T^* = [0, t^*)$ this interval such that $d^{\circ}(t) < \delta$ for any $t \in [0, t^*)$, and for any positive and sufficiently small ε, $d^{\circ}(t^* + \varepsilon) \geq \delta$ with $d^{\circ}(t^*) = \delta$. This interval a subset of $T^{\delta} : \left\{ t \in \Re^+ | d^{\circ}(t) < \delta \right\}$ and it represents the biggest time interval within which the Short-Run Endogenous CP holds and, therefore, it also represents our *short run*.

For a given value of the threshold, the Short-Run Endogenous CP-clause may be considered all the less restrictive the longer T^* is. In the best case ever, T^* has no upper bound, $T^* = T^{\delta} = \Re^+$, and the Short-Run Endogenous CP condition always holds.

One of the weaknesses of this approach relies on the choice of an appropriate threshold. First of all, notice that the end of the short-run, (the value t^*) is not invariant to changes in the unit of measure and therefore the choice of the threshold must take this fact into account. Finally, since this problem seems to be properly understood by using a continuum framework,

[26] See the second section of the technical appendix.

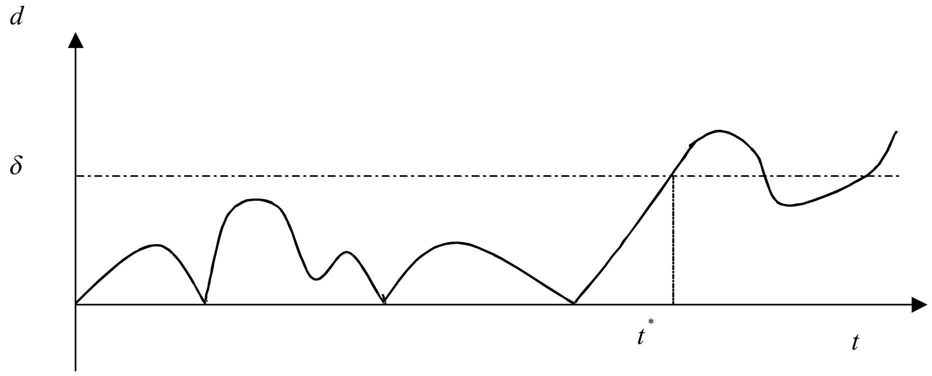

Figure 2: The distance function and the interval T^*.

it is always difficult to choose a sharp boundary between the two cases. But this is part of a more general problem.

Summing up, by making use of this modified dynamic system, Marshall implicitly introduces what we have defined as the short-run endogenous CP-clause according to which, with reference to the initial steady state taken into account, the slow variables are assumed to be stationary despite the movement of the fast ones. What happens when time horizons get wider, that is, for $t > t^*$?

7 Long-run Endogenous CP

7.1 Effective and modified flow

Sooner or later there comes a time in which the effective dynamics can no longer be well approximated by the modified dynamics. More formally, for any $t > t^*$, the difference between $x(t)$ and $\tilde{x}(t)$ is non-negligible and therefore, by definition, no short-run endogenous CP-clause can be successfully invoked in the long run[27].

However, as we have already anticipated, Marshall suggests a way out of this impasse in order to restrict the number of variables to be taken into account in the long run. He argues that "...while market prices oscillate about

[27] By definition, in the long run, all the components of s approach their new equilibrium values and, therefore, the vector α, which is part of s, cannot be considered to be in steady state. Hence, when the influence of α on x is notable (but otherwise there would be no reason at all for them to be taken into account in the analysis), the Short-run Endogenous CP does not hold any longer.

a position of market equilibrium, which perhaps oscillate about a position of short period equilibrium, that position in its turn may not remain stationary, but may move onwards in one direction, or may oscillate more slowly round a position of long-period normal equilibrium"[28]. Marshall's arguments seems to resemble that of the so-called *moving equilibrium method*[29]: if we accept the assumption that the fast variables approach rather quickly to their equilibrium, we are entitled, when the time horizon is long enough, to approximate the true movement of the fast variables to the movement of their equilibrium values. As a consequence, we can describe the whole $(m+n)-$dimensional dynamic system by concentrating on the dynamic behaviour of just n variables, the slow ones. This happens because the partial equilibrium values of the fast variables, in the proximity of which they are assumed to get trapped, are indeed a function of the particular values taken by the slow variables; hence, they can be deduced by the actual values assumed by the slow variables, which need not be in equilibrium. Again, the steady state for the dynamic system must be unique and stable, as Marshall seems to believe.

We can formalize this intuition as follows. We consider the set $F = \{(x,\alpha) : f(x,\alpha) = 0\}$. This set is a differentiable $m-$dimensional manifold in $\Re^{n+m}$ if the matrix $Df(x,\alpha)$ has maximum rank (n) for every $(x,\alpha) \in F$. This requires that for every point in F there is at least one square submatrix $(n \times n)$ in Df having non zero determinant. Therefore, there is always at least one component of s which can compensate for the movement of another, thereby allowing for the trajectories of the system to remain in F. By the implicit function theorem, from $f(x,\alpha) = 0$ it is therefore possible to locally represent the partial equilibrium value of x as a function of the value assumed by α, that is, $x = \bar{x}(\alpha)$ for $(x,\alpha) \in F$.

In general F, as with G before, is not a positively invariant, therefore for every $s \in F$, $Df(s)z(s) \neq 0$ holds except for a set with a zero Lebesgue-measure. That is, for "almost" every $s \in F$, the vector field $z(s)$ does not belong to the tangent space of s, $T_sF := \{\partial s \in \Re^{n+m} : Df(s)\partial s = 0\}$, representing the set of directions towards which the system can approach maintaining the constraint $f(s) = 0$. In other words, moving from an arbitrary $s^{\circ} \in F$ towards the direction determined by $z(s)$, the velocity of x leaves its zero value with probability one and therefore the trajectories of the system abandon F.

However, whenever for $s \in F$, the vector field $z(s) = (f(s), g(s)) = (0, g(s))$ is not tangent to F, it is possible to define a new vector field tangent

[28] Marshall (1961, vol.II, p.395).

[29] For an extended treatment of the "moving equilibrium method", see Samuelson (1947) and Schlicht (1985).

to F. Consider the expression $Df(s)z(s) = 0$ the geometrical interpretation of which is that, for each component x_i and for every $s \in F$, the gradient of f_i must be orthogonal to the field vector $z(s)$. Since the gradient of f_i is orthogonal for every i to the manifold F itself, then the field vector z is tangent to F. When z is not tangent to F in s ($Df(s)z(s) \neq 0$), then at least a component x_i exists such that Df_i is not orthogonal to $z(s)$. The orthogonal projection of $z(s)$ over T_sF is instead orthogonal to Df_i and therefore it is tangent to the manifold.

The orthogonal projection of z, which is defined over the tangent bundle of the F manifold ($TF = \bigcup_{s \in F} T_sF$), is univocally obtained by subtracting from z a particular sum of vectors representing quantities μ of row vectors of Df. Since these quantities vary as we change position on the F manifold, they can be considered as functions of the several points belonging to the manifold. Therefore $\mu = \mu(s)$ where $\mu(s)$ is a $(n \times 1)$ vector and, as usual, $s \in F$.

In any point $s \in F$, the component of z which is orthogonal to F is given by $\mu(s)Df(s)$ and therefore the expression of the modified field tangent to F is given by

$$\begin{aligned} \check{z}(s) &= z(s) - Df(s)^T \mu(s) \\ \check{z} &: F \rightarrow TF \end{aligned} \tag{24}$$

The field $\check{z}(s)$ is tangent to F and the orbits it generates completely lie on F. Therefore $\check{z}(s) = z(s) - Df(s)^T\mu(s) \in TF$ and $Df(s)\check{z}(s) = 0$. From this last expression we can obtain the value of $\mu(s)$ From (24) in fact we have $Df(s)\left[z(s) - Df(s)^T\mu(s)\right] = 0$ and finally

$$\mu(s) = \frac{Df(s)z(s)}{[Df(s)]^2} \tag{25}$$

Substituting this value into the modified tangent field we have

$$\check{z}(s) = z(s) - Df(s)^T \frac{Df(s)z(s)}{[Df(s)]^2} \tag{26}$$

where $\check{z}(s) = \left[\check{f}(s), \check{g}(s)\right]$ e $s \in F$.

7.2 The negligibility condition

"Inventing" a modified field for the whole system is certainly a useful device since it allows us to reduce the number of independent variables in the

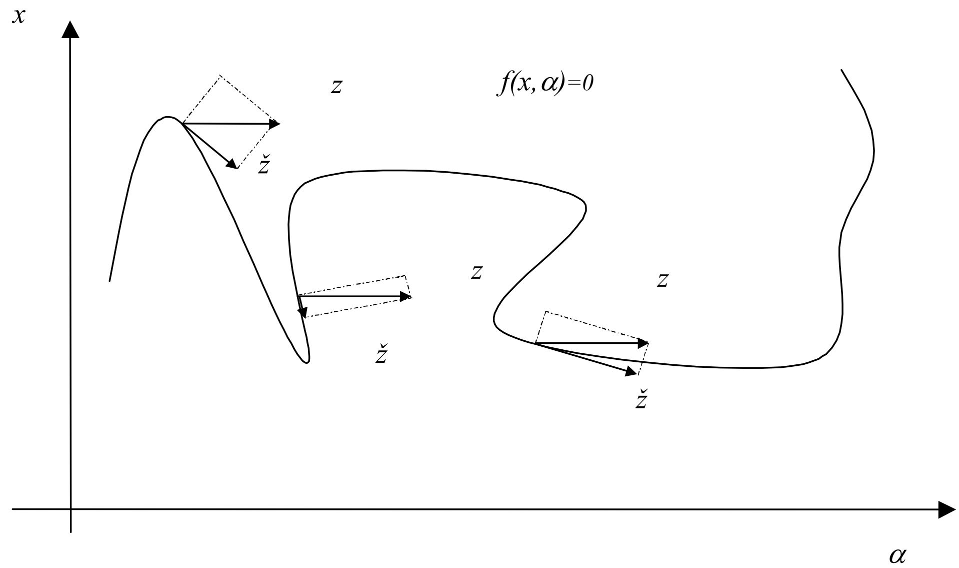

Figure 3: The modified vector field with $n = m = 1$.

system. We can then deduce the dynamics of $n+m$ variables just by looking at m variables.

$$\begin{aligned} \dot{x} &= \check{f}(\bar{x}(\alpha), \alpha) = \check{f}(\alpha) \\ \dot{y} &= \check{g}(\bar{x}(\alpha), \alpha) = \check{g}(\alpha) \end{aligned} \tag{27}$$

But does it make any sense? How do we justify the modification of the field vector in F?

We can reasonably introduce this change in the field if there is a $t^{**} \in \Re^+$, function of the initial state $s(0)$, such that the effective trajectories are defined and remain so "close" to F that they can be considered to lie entirely on F. If this is the case, we are allowed to use $\check{z}$ (tangent to F) instead of z and make "as if" the $x's$ are always in a partial (but moving) equilibrium state.

Once again, in order to represent the idea of "vicinity", we define an open neighbourhood $U(F)$ with centre in F such that for any $(x, \alpha) \in U(F)$ the difference between (x, α) and $(\bar{x}(\alpha), \alpha) \in F$ is not relevant with respect to the context and the aim of the analysis. Since $\bar{x}(\alpha)$ is such that $f(\bar{x}(\alpha), \alpha) = 0$, then if $(x, \alpha) \in U(F)$ the $x's$ are "substantially" in equilibrium.

The substitution of the field is therefore appropriate if z is such that each trajectory entering $U(F)$ gets trapped in it, that is, $U(F)$ is positively invariant. Since the steady state belong to F, and since $U(F)$ has the same dimension of the state space, if the system is stable (coherently with Marshall

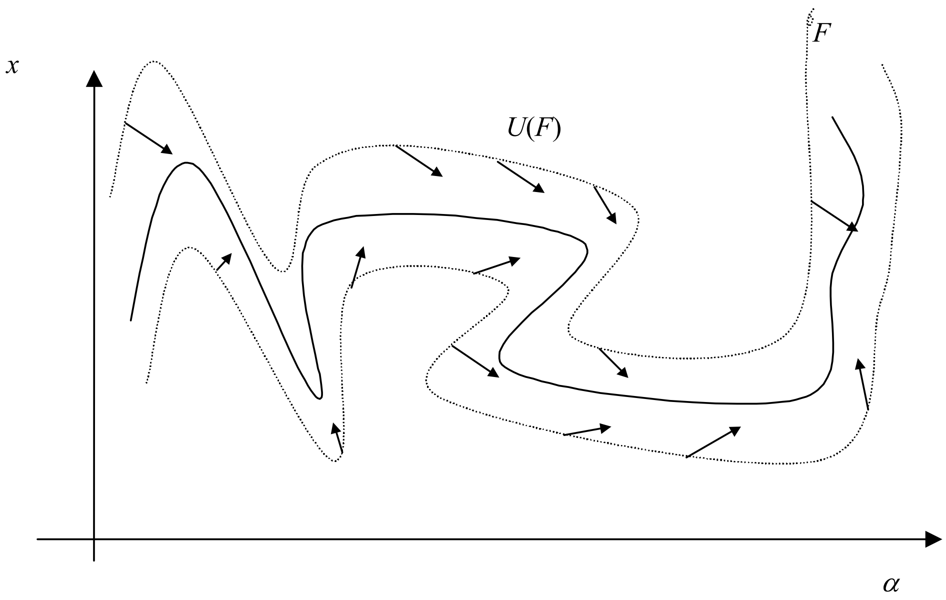

Figure 4: The positively invariant neighbourhood $U(F)$.

and, for example, with neoclassical growth theory) there is always a $t^{**} \in \Re^{+}$, function of s°, such that, if $U(F)$ is invariant, then $\varphi(t, s^{\circ}) \in U(F)$ for each $t \geq t^{**}$. In our framework, t^{**} is when the long run begins.

We can give the following

Definition 3 *(Long run Endogenous CP) Given a vector of initial states* $s(0)$, *the long run endogenous CP is verified with respect to a neighbourhood* $U(F)$ *and for* $t \geq t^{**}$ *, if* $\varphi(t, s^{\circ}) \in U(F)$ *for any* $t \geq t^{**}$ *.*

An implicit assumption in Marshall's time period analysis seems to be the following: the $x's$ are so fast that we can assume that they approach their equilibrium values when the dynamics of the vector α are still negligible with respect to δ (that is before t^{*}) and they remain "close" to their equilibria manifold F as α start to move in a "perceptible" way. In other words, $t^{**} \leq t^{*}$.

So we can use the field $\check{z}$ and make "as if" the trajectories lie entirely on F which is only $m-$dimensional while the state space is $n+m$-dimensional. In other words: after t^{**} the dynamics of α is such that x never hit the target (remain on F) but they never get too far from it[30].

Thanks to this analytical device, Marshall is actually able - still in the long run - to approximate the behaviour of the whole economic system by focusing

[30] Since a concept of threshold is involved also here, even this notion of CP may suffer

on one if its parts. But unlike the short-run endogenous CP, in this case no variable is assumed to be frozen. The crucial assumption, which allows for this simplification, is that the fast variables are constantly in (temporary) equilibrium, but this equilibrium is moving since its value is a function of the (moving) state of the slow variables. Considering this device as another version of the CP-clause is a matter of definition: on the one hand the long run endogenous CP involves a simplification because it reduces the number of variables to be taken into account; on the other hand, in this case CP cannot be translated as "other thing being equal" because, actually, "other things are *not* equal".

8 Conclusion

We have made an attempt to clarify and formalise the several meanings of *ceteris paribus* contained in Marshall's work. This aim is pursued by means of the analysis of a multivariate dynamic system which is conceived in such a way as to embody all the relevant features of Marshall's view of the economic system. The CP-clause is interpreted with a dynamic approach and its several meanings corresponds to different characteristics of the flow function. The analysis is developed taking into account both the overlapping of historical changes and the pure adjustment process following a single shock. In the latter case, the element of time plays a crucial role by mitigating the intensity of the feedbacks among the several variables and by allowing for a time-period analysis which is strongly linked to the concept of ceteris paribus. The basic result is a double distinction (Exogenous and Endogenous CP and, within the latter, Short-run and Long-run CP), which we believe might be useful to properly understanding Marshall's view of the economic system. In particular, since the concept of Long-run Endogenous CP, does not require that the set of variables involved in it be "frozen", it represents a generalization of the "classic" notion.

from the same weakness as the previous one. If we define $U(F)$ as

$$U(F) = \left\{(x, \alpha) \in X \times \Gamma : d\left[(x, \alpha), (\bar{x}(\alpha), \alpha)\right] < \lambda\right\}$$

where d is the eucledian distance function and $\bar{x}(y), y) \in F$, then it might not be so easy to find such a $\lambda \in \Re^{+}$. $U(F)$ must, in fact, be small enough for λ to be negligible but big enough in order to be positively invariant.

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9 Appendix

9.1 Conditions for G to be positively invariant

$G = g^{-1}(0)$ is the pre-image of $0 \in \Re^m$ according to $g : \Re^{m+n} \to \Re^m$ and it identifies each pair (x, α) associated to a zero value of the vector field g for the vector α. The theorem of the implicit function tells us that, if for any point of G the jacobian matrix of g has maximum rank, then the set G (if not empty) is a C^1 $n-$ dimensional *differentiable manifold*[31]. This assumption guarantees that if any variable in s changes its state, there is always at least another variable able to change its state in order to constantly verify the condition $g(x, \alpha) = 0$. In other words, a *compensatory* variable always exists in G.

Once it is assumed that G is a C^1 manifold, it is still not guaranteed that all the trajectories entering G do not leave it as $t \to \infty$. In order for G to be positively invariant, we need that

$$Dg(s) \cdot z(s) = 0 \tag{28}$$

In other words, the field z must belong to the tangent space of G, $T_sG := \{\partial s \in \Re\}$. If, for instance, the $m+n$ variables represent the market quantities of an economy, the condition $z \in T_sG$ states that, if the initial value is an equilibrium for the vector α, the economy moves along G according to the vector field preserving the equilibrium in those m markets. The CP-clause will then be guaranteed for those m markets.

The condition (28) could be also written as

$$D_xg(x, \alpha) \cdot f(x, \alpha) + D_xg(x, \alpha) \cdot g(x, \alpha) = 0$$

Since $g(x, \alpha) = 0$ for any $(x, \alpha) \in G$, then the direction of the vector field for any $(x, \alpha) \in G$ is parallel to the n-dimensional hyperplane $\{(x, \alpha) \in X \times Y : \alpha_j = 0, \forall j \in M\}$. Then the condition reduces to

$$D_xg(x, \alpha) \cdot f(x, \alpha) = 0 \tag{29}$$

In other words, only those variables belonging to the x vector are allowed to be compensatory variables. If starting from an initial point $(x^{\circ}, \alpha^{\circ}) \in G$

[31]It is worth discussing the conditions that guarantee that G is a differentiable manifold. According to the Sard theorem, the set of the critical values of a differentiable map $g : X \to Y$ has zero Lebesgue-measure. The main message of this theorem is the following: consider the map $g : S \to \Re^m$, a given $(x, \alpha) \in G$ could be such that $(D_xg, D_\alpha g) \neq (0, 0)$ - and hence (x, α) is a *regular* point, or it could be such that $(D_xg, D_\alpha g) = (0, 0)$. In the latter case (x, α) is a *critical* point. It is always o find a neighborhood of a regular point such that the function is monotone and therefore it can be locally represented using a cartesian graph. This is why we can always apply the implicit function and obtain the inverse map.

By contrast, the function g is never one-to-one any neighbourhood of a critical point and therefore we can't obtain the inverse function $g^{-1}(x, \alpha)$ and we can't say whether G is a differentiable manifold or not.

Unfortunately, it is quite likely to incur in a critical value. In order for this to happen, we just need the g to be defined in a compact set (Weierstrass theorem). However, the Sard theorem states that the critical values of g belong to a zero-measure set so that, starting from any initial $(x, \alpha) \in G$, we will find a critical value with probability zero. This seems to be enough for our approach.

we want the trajectories to remain in G, only the x vector can change in order to compensate for a movement of another variable which, otherwise, will take us away from G.

As a consequence, if G is positively invariant, any set

$$G(\alpha^{\circ}) = G \cap \{(x,\alpha) \in X \times Y : \alpha_j = \alpha_j^{\circ}, \forall j \in M\} = \{(x,\alpha^{\circ}) : g(x,\alpha^{\circ}) = 0\}$$

will also be positively invariant.

Now, in order for the set $G(\alpha^{\circ})$ to be a differentiable (sub-)manifold itself, it is necessary that the $(m \times n)$ submatrix $D_x g$ has maximum rank in any point such that $g(x,\alpha^{\circ}) = 0$. This will guarantee that in any point of $G(\alpha^{\circ})$ there is a compensatory variable contained in the vector x.

The dimension of $G(\alpha^{\circ})$ is not the same as G. The latter is $n-$dimensional and the loss of m dimension with respect to the state space is due to the fact that m constraints $g_j(x,\alpha)$ had to be verified and they subtract m degrees of freedom to the joint movement of x and α. Now we have other m constraints associated to the fact that the m variables contained in α cannot change their state and therefore, if G is invariant, all the trajectories starting from G will also lie on the hyperplane $\{(x,\alpha) \in X \times Y : \alpha_j = \alpha_j^{\circ}, \forall j \in M\}$.

The dimension of $G(y^{\circ})$ would then be $(n-m)$.

The condition (29) is apparently less restrictive than the (19) we have found above since it does not require a constant zero value for the partial derivatives of g. We would then be not obliged to assume the interruption of the feedbacks among the several variables in order for the G to be invariant. However, the latter condition becomes *necessary* whenever $m \geq n$. If $m \geq n$, the n equations $D_x g(x,\alpha) \cdot f_i(x,\alpha) = 0$ cannot be linearly independent; the set $G(\alpha^{\circ})$ then is not a manifold and the only way for the trajectories to start from and remain in G is for the slow variables α not to be affected by the change in the state of the fast variables x. A quick and easy check of this conclusion can be made by simply considering the case $n = m = 1$.

9.2 Existence of the maximum interval

We first show that the set $T^{\delta} := \{t \in \Re^{+} | d^{\circ}(t) < \delta\}$ has a non-empty interior part. Since both $x(t)$ and $\tilde{x}(t)$ are continuous functions in t, also $d^{\circ}(t)$, which is the norm of a difference between two continuous functions, is a continuous function too. Since $d^{\circ}(0) = 0$ and given the tolerance threshold $\delta \in \Re^{+}$, there is always an $\epsilon \in \Re^{+}$ as small as we like, such that $d^{\circ}(t) < \delta$ for any $t \in \{t \in \Re^{+} : t < \epsilon\}$. This set has a non-empty interior part. Since $T^{\delta} \subseteq t \in \{t \in \Re^{+} : t < \epsilon\}$ we have $\overset{\circ}{T^{\delta}} \neq \varnothing$.

The inferior extremum of this set is $t = 0$ while the superior extremum, in case T^δ is upper-bounded, is that instant of time t^δ such that $d(t) > 0$ for any $t \geq t^\delta$. If T^δ is not upper-bounded, then $\sup T^\delta = +\infty$.

We then show that the set T^δ always contains an initial interval $T' = [0, t')$ such that $\overset{\circ}{d}(t) < \delta$ for any $t \in [0, t')$. The set T^δ is the pre-image of an open set in the target of the function $\overset{\circ}{d}$. This pre-image, in the topology induced by $\Re$ on $\Re^+$, is open because $\overset{\circ}{d}$ is continuous. Therefore, T^δ is an open set. But an open set in $\Re^+$ is made of a countable set of open intervals, so that we are certain that there is always an interval of the kind $T' = [0, t')$.

This result is useful because we can't rule out the possibility that $\overset{\circ}{d}(t)$ is non-monotone and therefore the set T^δ could well be a disjointed set. But, in order to bound the field of action for the short-run endogenous CP-clause, we are not interested in T^δ but in the initial maximal interval contained in it. This interval coincides with the maximum extension for the interval $T' = [0, t')$.

In order to formalize these intuitions let's consider the family of intervals of the kind $T = [0, t)$ contained in T^δ, given by

$$K := \left\{ T \subseteq T^\delta : \overset{\circ}{d}(t) < \delta, \forall t \in T \right\}.$$

K has a non-empty interior part and it is upper-bounded by the largest interval contained in T^δ. The interval $T' = [0, t')$ is just one of these.

We now consider the set of superior extremum for the intervals contained in K. We call this set S

$$S := \left\{ t \in \Re^+ : t = \sup_{T \in K} T \right\}$$

If S is upper-bounded, then the upper-bound is a t^* such that the distance function reaches the threshold δ *for the first time.* This is the value of t (the end of the short-run) we are looking for

$$t^* = \sup S$$

Obviously, we have $\overset{\circ}{d}(t^*) = \delta$ and $d(t) < \delta$ for any $t \in [0, t^*)$. If S is not upper-bounded then $t^* = +\infty$.