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**FEW BAD APPLES OR PLENTY OF LEMONS:
WHICH MAKES IT HARDER TO MARKET PLUMS?**

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Few bad apples or plenty of lemons: which makes it harder to market plums?*

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Abstract

We analyse a competitive commodity market with a large number of buyers and sellers where: a. Individual qualities, either high or low, are not observable by buyers; b. Sellers strategically announce prices and buyers decide whether to buy having observed sellers' actions. We find that the set of robust equilibria includes only fully separating equilibria. In any robust equilibrium the low quality is always traded. The high quality is traded if demand is sufficiently strong, so that low quality sellers are unable to satisfy all buyers, and is never traded otherwise. Hence, few rotten apples is better than a plentiful of lemons for plums' sellers.

JEL CODES: D4, D8, L15

KEYWORDS: Market for lemons, Adverse Selection, D1, price-setting, off-equilibrium beliefs

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1   Introduction

We study a competitive market where sellers strategically take their pricing decisions and buyers, who are uninformed about the quality of the commodity in the hands of individual sellers, act after observing sellers' actions. Akerlof (1970) showed that in a market where buyers are less informed than sellers, the presence of lemons might drive the best qualities out of the market. The ultimate objective of our analysis is to understand how this (possibly adverse) selection mechanism operates when sellers' pricing decisions might convey information relevant to the buyers.

We consider a commodity market with a large number of sellers and buyers. Sellers fall into two categories depending on whether the quality they are endowed with is high or low. Buyers know the proportions of low and high qualities in the market but cannot observe individual qualities. Sellers are informed about their own quality and have the same information as the buyers regarding other sellers. Sellers move first and simultaneously decide which price(s) to announce, if any. Buyers decide whether to buy and if so at which price(s), after having observed the actions of the individual sellers. Hence, the sequence of events is that of a signaling game where the actions of the informed might convey relevant information to the uninformed. At the same time, the market is competitive to the extent that there is a large number of buyers, who are price takers, and a large number of sellers, who engage in price competition.

When buyers' off-equilibrium beliefs are restricted on the basis of the D1 dominance criterion (see Cho and Kreps 1987), the resulting set of robust equilibria only includes fully separating equilibria, where sellers of different types undertake different actions, thereby fully revealing their type. In any of these equilibria, the low quality is always traded. Whether the high quality is traded or not depends, other things equal, upon market conditions. More precisely, if the supply of low quality is enough to satisfy all buyers, only the low quality is traded. Viceversa, if market conditions are reversed, both the high quality and the low quality are traded. Plums are easier to market if the Akerlof's selection mechanism is induced by the presence of few rotten apples rather than a plentiful of lemons.

The intuition behind this novel result goes as follows. In any equilibrium where only low quality is traded, if the supply of low quality is not enough to satisfy all buyers, low quality sellers sell their commodity with probability one at a price equal to buyers' reservation price, thereby making strictly positive profits. Differently, the high quality sellers, who are not willing to sell at the market price, make zero profits. Hence, only the low quality sellers face an opportunity cost of deviating from their equilibrium strategy. Because of that, a low quality seller would find it profitable to deviate by announcing a higher price, p, if and only if the probability to sell at p is high enough. High quality sellers, instead, profit from such a deviation so long as there is a positive probability to sell. This probability could always be so so low that only high quality sellers could possibly benefit from deviating. But then, buyers would regard

such a deviation as a signal of high quality. We show that there always deviations, p, such that a buyer prefers to buy at p, given the refined beliefs associated with p. This creates incentives for any high quality seller to deviate, which destroys the equilibrium.

The above reasoning does not hold when market conditions are reversed so that the supply of low quality is enough to satisfy all customers. Under these circumstances, in any equilibrium where only the low quality is traded, the competitive pressure ensures that the price equals sellers' reservation utility, so that low quality sellers make zero profits as well as the high quality sellers who are not selling their commodities. Therefore, neither type of seller faces any opportunity cost of deviating. Hence, off-equilibrium prices do not provide any meaningful signal, and the equilibrium where only the low quality is traded is robust.

When the supply of low quality is not sufficient to satisfy the demand, a robust separating equilibrium where both qualities are traded arises, and only then. In this equilibrium the high quality trades at a higher price than the low quality. Sellers are on the long side of the market at the higher price and are never on the long side at the lower price. Hence, sellers charging the lower price are surely able to sell their commodity, while the probability to make a sale at the higher price is less than one. This explains why a fully separating equilibrium where both qualities are traded is sustainable: the comparatively low chances to sell at the higher price dissuade low quality sellers from announcing it. Since all buyers are sure to get a commodity, and one of high quality, if buying at the higher price, the only viable deviation would be to announce a price (slightly) lower than that price. However, so long as the probability to sell at the higher equilibrium price is such that low quality sellers are indifferent between charging either the low or the high equilibrium price, both types of sellers face the same incentives to deviate. Therefore, deviations do not have any signaling content. This explains why an equilibrium where both qualities are traded is robust.

The same logics also explains why the amount of high quality being traded is less than the total supply of high quality. Incentive compatibility for low quality sellers dictates that the probability to make a sale at the high price should stay lower than one. This implies that, even if demand exceeds total supply, there will always be sellers (of high quality) unable to sell their commodity. This, even when they strictly prefer to do so. Prices should fall to equate demand and supply, but they fail to do so because of the imperfect information. High quality sellers are therefore rationed, in a sense similar to the concept of rationing put forward by Stiglitz and Weiss (1981). Moreover, the more buyers value quality, the higher is the price at which the high quality is traded. The higher the price of high quality, the lower must be the probability to sell at that price, which means that less of the high quality is traded. The model thus displays a sort of high quality's curse.

The paper relates to the classic literature on the market for lemons. In his research upon the nature of equilibrium in market with adverse selection, Wilson (1980) con-

siders, among various price setting conventions, a version of Akerlof's model with price setting sellers. He finds equilibria in which sellers of different qualities announce different prices. Different prices can be sustained in equilibrium as long as sellers are on the long side of the market at the relatively higher prices. These equilibria, however, do not exhaust the set of possible equilibria, which also includes single price (pooling) equilibria and other hybrids. As Wilson explains, the reason why the equilibrium can take so many forms, is

> "the absence of restrictions on the expectations of agents outside the set of [equilibrium] prices actually announced" [page 126].

He also comments:

> "we would like the equilibrium to have the [robustness] property that even if sellers occasionally experiment by announcing a new price, their experience will never lead them to revise their expectations in such a way that they permanently alter their equilibrium behaviour. [..] The problem is that it is no longer obvious what it means to restrict expectations to be 'correct' at prices at which no trade takes place" [page 127].

Our setup, in which buyers observe individual sellers' actions, provides solid grounds for applying restrictions to the off-equilibrium beliefs based on various dominance criteria, thereby reducing the typical indeterminacy to a great extent. This enables us to fully characterise the set of (robust) equilibria. Laffont and Maskin (1986) study an oligopolistic market where sellers can signal quality through their prices and conclude that, even in the case of just two firms, the problem of characterising all perfect Bayesian equilibria seems intractable.

The reduced indeterminacy makes it also possible a welfare comparison between the market with price setting sellers and a centralized Walrasian market, where agents are price-takers and trade occurs at a single price set by an auctioneer. We find that the two models yield the same result only if the supply of low quality is enough to satisfy the demand. Otherwise, the Walrasian model can either do better or worse than the model with strategic price setting. This depends on how strong is the demand compared to total supply and on whether the welfare gains from trading the low quality exceed those from trading the high quality.

In a series of papers, Grossman and Stiglitz[1] discuss how the informational role of equilibrium prices alters the concepts of competitive equilibrium and efficient markets. Rothschild and Stiglitz (1976) have shown that the informational role of contractual offers made by uninformed parties might compromise the existence of a competitive equilibrium. Our analysis shows how the information conveyed by off equilibrium

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[1]See Grossman and Stiglitz (1980, 1976).

prices announced by the informed can drastically reduce the number of plausible equilibrium outcomes.

Janssen and Roy (2002) study the lemons problem in the context of a dynamic model of a market for durable goods, where sellers are on the short side of the market. Low quality sellers have a higher incentive to sell their goods in the short run than high quality sellers, which generates a separating equilibrium where low qualities are sold earlier than high qualities.

From a technical viewpoint, we show that the equilibria robust to D1 satisfy a quasi-uniqueness property: all robust equilibria yield the same prediction in terms of traded qualities and associated prices and quantities. This result is in line with Cho and Sobel (1990). They show that, if a specific sorting condition is satisfied, D1 gives uniqueness and eliminates all types of equilibria that involve poolings of different types, unless the pooling occurs at the upper bound of the action space. In our model, the sorting condition is only satisfied for a subset of the action set.[2] Still, the robust equilibria are always fully separating and the equilibrium outcome is unique in terms of traded quantities and qualities.

The paper is organised as follows. In section two we present the model's structure. Section three discusses the equilibrium concept. In sections four, five and six we analyse the robustness of the various types of equilibria. Section seven explains the relationship between market conditions and the quality and quantity of trade. Section eight discusses the efficiency implications of the model. Section nine generalizes the results to the case of any finite discrete number of qualities. A final section concludes the paper.

2 The Model

We consider a competitive commodity market populated by a large number of buyers, B, and a large number of sellers, S. The set $\mathcal{S}$ of sellers is indexed by $s = 1, ..., S$; $s \in \mathcal{S}$. Each seller is endowed with one unit of commodity. Commodities come in two different qualities, $q = h, l$. Each seller has a probability λ to be endowed with quality h and probability $1 - \lambda$ to be endowed with quality l. We refer to sellers endowed with quality q as sellers of type-q. The law of large number applies, so that λ is interpreted as the fraction of type-h sellers. The individual monetary utility that type-q sellers derive from their commodity is $v(q)$, with $v(h) > v(l)$. $v(q)$ represents the reservation price for sellers of type q. Accordingly, a seller of type-q who sells at a price p receives a net payoff $p - v(q)$.

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[2] We note that Laffont and Maskin (1987) study the signaling effect of pricing decision in a monopolistic market. They restrict the set of perfect bayesian equilibria to those which maximize the monopolist's profits. In some cases, these could be pooling equilibria. However, these need not be robust to D1 as Cho and Sobel (1990) observe.

The set $\mathcal{B}$ of buyers is indexed by $b = 1, ..., B$; $b \in \mathcal{B}$. Each buyer consumes either one unit of commodity or nothing. Buyers share identical preferences defined by the monetary utility function $u(q)$, where $u(h) > u(l)$. We impose $u(q) > v(q)$ for $q = h, l$, which implies that under full information there are always gains from trade to be realized. For expositional purposes, we also impose $u(l) < v(h)$. In words, buyers are never willing to buy a low quality commodity at any price that is feasible for a high quality seller.[3] The distribution of qualities is common knowledge. However, buyers cannot observe individual qualities. Moreover, quality is not verifiable ex post.

The market functions as follows. Sellers act first. At stage 0, they simultaneously decide which action to play, while buyers do nothing. Each individual seller s plays an action a_s^0 that is either announcing a price $p \in \mathbb{R}^+$, in which case $a_s^0 = p$, or stay off the market, which corresponds to not announcing any price and is denoted by $a_s^0 = n$. For each individual seller, s, we denote $\alpha_s^0 : \mathcal{A}_s \to [0, 1]$ a probability distribution over the set of possible actions $\mathcal{A}_s = \mathbb{R}^+ \cup \{n\}$, and with $\mathbb{A}_s$ the set of all probability distributions over $\mathcal{A}_s$. We define $a^0 \equiv \{a_1^0, ..., a_S^0\}$, with $a_s^0 \in \mathcal{A}_s$, as an action profile for the sellers, where $\mathbf{p} \subseteq a^0$ is the set of prices associated with a^0. Note that $\mathbf{p} = \emptyset$ if the action profile a^0 is such that none of the sellers announces a price. We also define $\alpha^0 \equiv \{\alpha_1^0, ..., \alpha_S^0\}$, with $\alpha_s^0 \in \mathbb{A}_s$, as a mixed action profile for the sellers.

At stage 1, given sellers' actions resulting from stage zero, buyers simultaneously decide at which price(s) to buy if any, while sellers do nothing. Buyers observe the entire action profile of sellers, a^0 as well as the action played by each single seller s, a_s^0. Hence, when faced with an individual seller s, buyers' information is given by a_s^0 and a_{-s}^0. Given a set of prices $\mathbf{p}$, define a_b^1 as the action for an individual buyer b. Each buyer b can either choose to buy at some $p \in \mathbf{p}$, in which case $a_b^1 = p$, or not to buy, in which case $a_b^1 = n$ and the buyer's surplus equals zero. Accordingly, the set of possible actions for any buyer b is $\mathcal{A}_b \equiv \mathbb{R}^+ \cup \{n\}$. Note that if no prices are posted, buyers can only choose not to buy.[4] We call $a^1 = \{a_1^1, ..., a_B^1\}$ an action profile for the buyers, where $a_b^1 \in \mathcal{A}_b$, $\forall b \in \mathcal{B}$. We denote with $\alpha^1 \equiv \{\alpha_1^1, ..., \alpha_B^1\}$, $\alpha_b^1 : \mathcal{A}_b \to [0, 1]$, a mixed action profile for the buyers.

Buyers and sellers match randomly. If at any price there is excess demand/supply, the purchase/sale is randomly assigned. The probability to make a sale at any price p is given by the minimum between one and the ratio of buyers to sellers willing to trade at p. Symmetrically, the probability to make a purchase at p is the minimum between one and the ratio of sellers to buyers willing to trade at that price.

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[3] We give a brief account of what happens when $u(l) \geq v(h)$ in section 7.

[4] We assume that choosing to buy at a price that has not been announced ($p \notin \mathbf{p}$) is equivalent to not buying. In other words, sellers are committed to sell at the prices they announced.

3 Equilibrium analysis

A seller of type q selling at a price p makes non-negative profits if and only if $p \geq v(q)$. Prior to observing sellers' actions, buyers' (ex ante) individual beliefs that an individual seller s is of type h are given by the probability λ. By contrast, upon observing the action a_s^0 played by a seller s, and given the sellers' action profile a^0, buyers' (ex post) beliefs that the seller s playing a_s^0 is of type q are given by the conditional probability function $\sigma(q|a_s^0, a_{-s}^0)$. Given the belief function $\sigma(.,.)$, the expected utility of a buyer who obtains the commodity at a price p, is

$$\mu(p, a^0, \sigma) \equiv u(h)\sigma(h|p, a^0) + u(l)\sigma(l|p, a^0), \tag{1}$$

where $\sigma(l|p, a^0) = 1 - \sigma(h|p, a^0)$ is the probability associated with the buyers' beliefs that the seller s is of type l. Accordingly, $\mu(p, a^0, \sigma)$ is the maximum willingness to pay of each individual buyer.

Given a^0, and a^1, let us denote with $J(p, a)$ the probability to make a sale when announcing price p, where $a \equiv \{a^0, a^1\}$. Then, for a given a, the payoff of a seller s of type q is

$$\pi_s(a_s^0, a_{-s}|q, \sigma) \equiv \begin{cases} J(p, a)[p - v(q)] & \text{if } a_s^0 = p \\ 0 & \text{if } a_s^0 = n. \end{cases} \tag{2}$$

Similarly, for any individual buyer b, let us denote with $K(p, a)$ the probability to buy when playing $a_b^1 = p$. Again, K depends both on the action profile of buyers, a^1, and sellers, a^0. Given a^0, the payoff of any individual buyer b playing a_b^1 is

$$\pi_b(a_b^1, a_{-b}^1|a^0, \sigma) \equiv \begin{cases} K(p, a)[\mu(p, a^0, \sigma) - p] & \text{if } a_b^1 = p \\ 0 & \text{if } a_b^1 = n. \end{cases} \tag{3}$$

Let $\alpha \equiv \{\alpha^0, \alpha^1\}$ be a mixed action profile for sellers and buyers. Then, we denote with $J(p, \alpha)$ the expected value of the probability to sell at a given p, evaluated by taking the expectation of $J(p, a)$ over the probability distribution associated with α. Given the realization a^0 of α^0, we denote with $K(p, \alpha^1, |a^0)$ the expected value of the probability to buy at p, evaluated by taking the expectation of $K(p, a)$, over the probability distribution associated with α^1. Consequently, when agents play mixed strategies, we denote with $\pi_s(\alpha_s^0, \alpha_{-s}|q, \sigma)$ and $\pi_b(\alpha_b^1, \alpha_{-b}^1|a^0, \sigma)$ the sellers' and buyers' expected payoffs, respectively. Then,

Definition 1. *A Perfect Bayesian Equilibrium (PBE) is a profile of actions $\alpha^* = \{\alpha^{0*}, \alpha^{1*}\}$ - where the realisation a^{0*} of α^{0*} contains the set of market prices $\mathbf{p}^*$ - and a belief function $\sigma^*(q|a_s^0, a_{-s}^0)$ common to all buyers, with $\sum_{q=h,l} \sigma^*(q|a_s^0, a_{-s}^0) = 1$, such that:*

i Buyers and sellers are playing their best responses:

$$a \quad \pi_s(\alpha_s^{0*}, \alpha_{-s}^* | q, \sigma^*) \geq \pi_s(\alpha_s^0, \alpha_{-s}^* | q, \sigma^*) \ \forall s \in \mathcal{S}, q = l, h, \ \alpha_s^0 \in \mathbb{A}_s;$$

$$b \quad \pi_b(\alpha_b^{1*}, \alpha_{-b}^{1*} | a^{0*}, \sigma^*) \geq \pi_b(\alpha_b^1, \alpha_{-b}^{1*} | a^{0*}, \sigma^*), \ \forall b \in \mathcal{B}, \ a^{0*}, \ \alpha_b^1 \in \mathbb{A}_b \ ;$$

ii Buyers' beliefs are derived from sellers' strategies using Bayes rule where possible;

iii For all s, a^0, $\hat{a}^0$, and $q = l, h$, buyers' beliefs satisfy $\sigma^*(q | a_s^0, a_{-s}^0) = \sigma^*(q | \hat{a}_s^0, \hat{a}_{-s}^0)$ if $a_s^0 = \hat{a}_s^0$.

Conditions i)-ii) do not require any explanation. Condition iii) implies that beliefs about seller s are independent from other sellers' actions, even in the presence of deviations (Fudenberg and Tirole, 1991, p. 332). This condition is motivated by the fact that sellers different from s have no information about s's type, that is not also available to the buyers.

The equilibrium also requires that agents correctly assess the probabilities to sell and buy at the equilibrium prices. We shall also impose that, when choosing their actions, agents correctly guess the probability to make a transaction at any other price.[5] If we were not to apply this condition, any form of price competition could disappear under the appropriate choice of beliefs about the likelihood to make a sale or a purchase at prices different from the equilibrium prices. This would result in a large amount of equilibria with little economic interest.[6]

Various types of equilibria are, in principle, possible. In particular we could have:

a Separating equilibria, in which, by definition, different seller-types take different actions. In turn, these equilibria divide into two types: Type I, where only low quality sellers are on the market, while high quality sellers are out of the market and Type II, where all types of sellers are on the market (they announce a price);

b Pooling equilibria, in which all qualities are traded at the same price;

c Partially separating (Hybrid) equilibria, in which poolings of the various qualities that differ in their composition are traded at different prices.

While both in pooling and in hybrid equilibria all types of sellers are on the market and announce a price, we can have fully separating equilibria where only low quality sellers are on the market (type I separating equilibria).[7] Equilibria in which the high

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[5]The likelihood to sell at any off-equilibrium price clearly depends on buyers' beliefs about the quality of the seller who announces such a price. Since our analysis will refine off-equilibrium beliefs, this probability will become identifiable.

[6]For instance, an unjustified fear of not being able to sell the good at a lower price would prevent sellers from undercutting their competitors even in a perfect information framework.

[7]Note that whenever high quality sellers are on the market, low quality sellers must also be on the market since at any price at which is profitable to sell high qualities is also profitable to sell low qualities.

quality is driven out of the market have received particular attention since Akerlof's seminal paper, given the dramatic market failure they might entail. We turn our attention to the robustness of such separating equilibria first, and then we proceed with the analysis of robustness for the various types of equilibria where all types of sellers announce a price.

4   Type I separating equilibria: Only low quality sellers are on the market

Definition 2. *A type I fully separating equilibrium (FSE) is a PBE where high quality sellers are off the market.*

We note that staying off-market is an equilibrium strategy for high quality sellers as long as the probability to sell at any price which guarantees them a non-negative expected payoff is zero, given buyers' equilibrium strategies.[8] Define $\theta \equiv B/S$, the overall ratio of buyers to sellers, then

Lemma 1. *There always exist FSE of type I. Any FSE of type I satisfies the following properties: i. It implies a single equilibrium price p^* such that $\sigma^*(h|p^*, a^{0*}) \equiv \sigma^*(h|p^*) = 0$; ii. $p^* = u(l)$ if $(1 - \lambda) \leq \theta$, and $p^* = v(l)$ otherwise.*

Proof. See appendix.

When $1 - \lambda > \theta$, sellers of low quality are the long side of the market. Therefore, they compete to sell their commodity, which drives profits to zero $(p^* = v(l))$. If, on the other hand, they are the short side $(1 - \lambda \leq \theta)$, there is no competitive pressure. Thus, sellers announce $p^* = u(l)$ which ensures strictly positive profits.

A natural question to ask is what kind of off-equilibrium beliefs support the existence of these FSE of type I. Off-equilibrium beliefs should be such that none of the sellers has incentive to deviate from her equilibrium strategy and announce a price p different from the equilibrium price. As explained in the proof of lemma 1, a possible set of beliefs which support type I equilibria is that which assigns $\sigma(h|p, a^{0*}) = 0$ to any seller announcing a price $p \neq p^*$.

The question we ask is how reasonable are the off-equilibrium beliefs which support type I FSE. To answer this question we proceed by defining as reasonable those sets of beliefs which satisfy some standard criteria to be identified. Then, we assess under which conditions FSE of type I are robust, in the sense that they are supported by reasonable off equilibrium beliefs.

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[8]However, this is not the only equilibrium strategy for type h sellers. In fact, they could be in the market posting some price $p_h \geq v(h)$ at which trade does not occur. As we shall see when analysing type II separating equilibria, this is actually, under some circumstances, a robust equilibrium.

In the spirit of equilibrium refinements in signaling games, the procedure we adopt to test the robustness of FSE of type I, which will be then applied also to FSE of type II as well as pooling and hybrid equilibria, goes as follows. First, off-equilibrium beliefs associated with some deviation are refined by individuating type-action pairs that are "less likely" to occur. Second, the best response of the individual buyer, given the deviation and the refined beliefs, is computed under the assumption that all other buyers stick to their equilibrium strategies.[9] Finally, given buyers' best response, we consider whether the deviation is profitable for the individual seller under the assumption that no other seller is deviating.

4.1 Restrictions on off-equilibrium beliefs and robustness of FSE of type I

In general, given buyers' beliefs $\sigma(q|p)$ associated with a deviation p (to be specified), a seller of quality q will strictly benefit from deviating if:

$$\pi_s^*(q) < J(p, \alpha)[p - v(q)], \tag{4}$$

where $\pi_s^*(q)$ is the seller's equilibrium payoff. The seller would be indifferent if equality holds or would strictly prefer not to deviate if the inequality is reversed.

Upon observing a deviation p, each buyer decides whether to buy at p, or at any of the old equilibrium prices, $\mathbf{p}^*$. Thus, each buyer compares the expected payoff at p^* with the expected payoff from buying at p. After the deviation, individual buyers' payoff when buying at any $p^* \in \mathbf{p}^*$ is

$$\pi_b(p^*, \alpha_{-b}^{1*}, |a^0, \sigma^*) = K(p^*, \alpha^{1*}|a^0)\{\sigma^*(h|p^*)[u(h) - p^*] + (1 - \sigma^*(h|p^*))[u(l) - p^*]\}. \tag{5}$$

Before we proceed with the refinements of the off equilibrium beliefs, it is worth to point out two potential problems in determining the value of (5). First, in principle, $K(p^*, \alpha^{1*}|a^0)$ may be different from any of its possible equilibrium values because (at least) one seller has deviated and to p. Therefore, a^0, which contains p, occurs with probability zero in equilibrium. Also, some buyers may have deviated as well and in this case α^1 would be different from α^{1*}. Because of that, it is not straightforward that the payoff from buying at p^* should be the same as in in equilibrium, i.e. in absence of deviations. However, in order to assess whether an equilibrium is robust or not, we focus on (a) buyer's best response upon observing an off equilibrium price, given that other buyers are still playing the equilibrium strategy; and (b) given buyers' best response, a seller's incentive to deviate, given that other sellers are still playing their equilibrium strategies. Accordingly, since we are looking at a market with a

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[9]This implies that, in determining the hypothetical best response, the probability to obtain the good at the new price is one.

large number of buyers and sellers, the deviations we are analysing, which involve one seller/buyer, should not affect the probability to sell/buy at the old prices. Thus, denoting with $K^*(p^*)$ and $J^*(p^*)$ the probabilities to buy and sell at some equilibrium price $p^* \in \mathbf{p}^*$ respectively, we maintain that these remain unchanged when agents contemplate the possibility to deviate. To strengthen the argument, note that the approach we are taking is conservative. In small markets or if more than one seller deviates simultaneously, $K(p^*, \alpha^{1*}|a^0)$ must me either equal or lower than $K^*(p^*)$ since, if anything, the number of sellers selling at p^* is lower following the deviation. Hence, the level of the payoff from buying at p^* is either equal or lower than it was in equilibrium.

A second problem refers to the beliefs to be assigned to the sellers who did not deviate. Since a^0 contains a price that is never announced in equilibrium, the equilibrium assigns probability zero to the realization a^0, which, in principle, might have an effect on the beliefs that buyers assign to sellers who stick to p^*. However, this problem is ruled out by the standard requirement stated in point iii) of definition 1. Since the seller who deviated does not possess any information about the quality of sellers who did not deviate that is not available also to the buyers, the deviation should not change buyers' beliefs about these sellers.

In short, the above discussion implies that, after the deviation p, buyers will still be able to obtain what was their expected equilibrium payoff before the deviation, as long as they keep playing their equilibrium strategies. Having said that, we now turn to the analysis of different refinements and their consequences for the robustness of type I FSE.

4.1.1 Perfect sequential equilibria

A possible approach is to define as reasonable beliefs those beliefs derived from updating rules that are *credible* in the sense put forward by Grossman and Perry (1986). Accordingly, we shall require that the beliefs following the observation of an off-equilibrium price satisfy the following restriction. Whenever there exists a set of qualities $Q \subseteq l, h$, such that if

1. Each type $q \in Q$ weakly benefits from the deviation if it is thought that a type $q \in Q$ deviated;

2. Types $q \notin Q$ weakly lose from the deviation if it is thought that a type $q \in Q$ has deviated;

then buyers should believe that a seller of quality $q \in Q$ has deviated.

Accordingly, we can then refine the concept of PBE by adding the credibility requirement for the beliefs updating rule to definition (1). Formally,

Definition 3. *A PBE is said to satisfy the sequential perfection restriction (SPR) if the belief function σ satisfies the following property. Let $Q \subseteq \{l, q\}$ be any set of qualities. For all Q such that, for some $p \notin \boldsymbol{p}^*$,*

$$\pi_s(p, a_{-s}|q, \sigma) \geq \pi_s^*(q) \; \forall q \in Q \tag{6}$$
$$\pi_s(p, a_{-s}|q, \sigma) \leq \pi_s^*(q) \; \forall q \notin Q \tag{7}$$

σ must satisfy

$$\sigma = \sigma(q|p) = \Pr(q)/\Pr(Q), \tag{8}$$

where $\Pr(.)$ is the prior and $\pi_s^*(q)$ is the equilibrium payoff for a seller of type q.

The effect of imposing such a restriction on the type I equilibria discussed in lemma (1) is made clear by the following result:

Lemma 2. *a. Given $(1 - \lambda) > \theta$, there exist FSE of type I that satisfy the SPR; b. Given $(1 - \lambda) \leq \theta$, there exist FSE of type I that satisfy the SPR if and only if $\lambda \leq \hat{\lambda} \equiv \frac{v(h) - u(l)}{u(h) - u(l)}$.*

Proof. See appendix.

This implies that the separating equilibrium where high quality sellers stay off the market because of asymmetric information satisfies the sequential perfection restriction if low quality sellers are the long side of the market.

When low quality sellers are instead the short side, such an equilibrium satisfies the requirement of sequential perfection only if the prior probability to sample a high quality is low enough. In this situation, all sellers have incentive to deviate to some $p \geq v(h)$ so that the only credible beliefs are $\sigma(l|p) = 1 - \lambda$ and $\sigma(h|p) = \lambda$. In other words, the credible beliefs upon observing a deviation are equal to the priors. So long as $\lambda > \hat{\lambda}$, buyers are willing to buy at p. Hence, the equilibrium will be deviated. Consider now both conditions simultaneously: $\lambda \geq 1 - \theta$ (type l are the short side), and $\lambda \leq \hat{\lambda}$ (robustness). If $1 - \theta > \hat{\lambda}$, for any prior λ implying that type l are the short side, there is no robust equilibrium of type I. To understand this implication, note that $\hat{\lambda}$ is increasing in $v(h) - u(l)$ and decreasing in $u(h) - u(l)$. Hence, if $v(h) - u(l)$ is small (announcing a price close to buyers' reservation price for the low quality is feasible for the high quality), or if the gap in terms of buyers' utility between the two qualities, $u(h) - u(l)$, is large, there is no robust equilibrium in which high quality sellers stay off-market. This means that whenever high quality sellers are keen to sell their commodities and buyers attach great value to quality, equilibria in which high quality sellers do not announce any price are not robust. Surprisingly, this discussion does not apply, so that a type I robust equilibrium always exists, if the low quality sellers outnumber the buyers. In this case competition among the low quality sellers becomes so fierce that eventually drives the high quality out of the market.

It is interesting to note that the robustness condition stated in lemma (2) for the case $1 - \lambda \leq \theta$ is equivalent to the condition for the existence of an equilibrium where all qualities are traded in the standard textbook's model of adverse selection, where trade takes place at a single price set by an auctioneer. The condition $\lambda > \hat{\lambda}$ in fact implies that there exists a price $p \geq v(h)$ at which buyers are willing to buy a pooling of the two qualities. We shall see in the next subsection that most of these intuitions hold true also for the Divinity criterion put forward by Banks and Sobel (1987).

However, one could ask whether $\sigma(h|p) = \lambda$ can be further restricted. Such a doubt is motivated by noting that in equilibria where $p^* = u(l)$, low quality sellers - who are making strictly positive profits - can be made worse or better off by deviating. By contrast, high quality sellers have nothing to lose, whatever the buyers' response. Thus, it seems compelling that, upon observing a deviation $p \geq v(h)$, sellers should assign a higher probability (than the prior) to high quality sellers.

This idea of comparing the opportunity costs is made somewhat more explicit in the refinements discussed in the next subsection.

4.1.2 Divine and D1-robust equilibria

A second approach, alternative to the one discussed in the previous subsection, is to follow Banks and Sobel (1987) notion of Divine Equilibria and Cho and Kreps (1987) version of their concept of Universal Divinity, known as D1. We remand to these works for a general discussion of these refinements in standard signaling games.

Consider an equilibrium and assume that an individual seller deviates and announces price p. If, at the off equilibrium price p, beliefs are given by σ, buyers' expected payoff from buying at p for a given profile of mixed actions α^1 is:

$$\pi_b(p, \alpha^1_{-b}|a^0, \sigma) = K(p, \alpha^1|a^0)\{\sigma(h|p)[u(h) - p] + (1 - \sigma(h|p))[u(l) - p]\}, \quad (9)$$

where a^0 comprises the set of prices $\mathbf{p}^*$ and the deviation p. Given some beliefs $\hat{\sigma}$, let

$$MBRP(\hat{\sigma}, a^0) \equiv \{\alpha^1 : \pi_b(\alpha^1_b, \alpha^1_{-b}|a^0, \hat{\sigma}) \geq \pi_b(\alpha^{1'}_b, \alpha^1_{-b}|a^0, \hat{\sigma}) \; \forall b \in \mathcal{B}, \alpha^{1'}_b \in \mathbb{A}_b\} \quad (10)$$

denote the set of *profiles* α^1 that are formed by best responses when beliefs at p are given by $\hat{\sigma}$. In other words, $\alpha^1 \in MBRP(\hat{\sigma}, a^0)$ is a Nash equilibrium of the subgame played by buyers when a^0 is realized and beliefs are given by $\hat{\sigma}$.[10] Define $\hat{\alpha}$ a profile of mixed actions that makes low quality sellers indifferent between deviating to p and announcing p^*. Let $J^l(p, \hat{\alpha})$ be the probability to sell associated with $\hat{\alpha}$, i.e., for $p > v(l)$,

$$\pi_s^*(l) = J^l(p, \hat{\alpha})[p - v(l)], \quad (11)$$

---

[10]Note that we consider profiles of mixed best responses because we have more than one receiver (buyer).

where $\pi_s^*(l)$ is the equilibrium payoff of the seller. Then, whenever the probability to sell when deviating, $J(p, \alpha)$, is greater than J^l, low quality sellers would always deviate. Similarly, high quality sellers would always deviate if

$$J(p, \alpha) > J^h, \tag{12}$$

where, for $p > v(h)$,[11]

$$\pi_s^*(h) = J^h(p, \tilde{\alpha})[p - v(h)] \tag{13}$$

and $\tilde{\alpha}$ is the profile that makes high quality sellers indifferent between the deviation and their equilibrium payoff. Based on equations (11) and (13) we define:

$$R_1 \equiv \bigcup_{\hat{\sigma}:\hat{\sigma}(h|p)+\hat{\sigma}(l|p)=1} \{\alpha^1 \in MBRP(\hat{\sigma}, a^0) : \ J(p, \alpha) \geq J^l\} \tag{14}$$

and

$$R_2 \equiv \bigcup_{\hat{\sigma}:\hat{\sigma}(h|p)+\hat{\sigma}(l|p)=1} \{\alpha^1 \in MBRP(\hat{\sigma}, a^0) : J(p, \alpha) > J^h\}. \tag{15}$$

According to Divinity, then, off equilibrium beliefs should satisfy the following restriction: whenever $R_1 \subset R_2$, $\sigma(h|p) \geq \lambda$ must hold. Thus, if, for some deviation p, high quality sellers have incentive to deviate in any situation in which low quality sellers have a weak incentive to deviate, the posterior probability that the seller is of high quality should not decrease after observing p. The next lemma gives necessary and sufficient conditions for $R_1 \subset R_2$.

Lemma 3. *Assume R_2 is non-empty. Then, $R_1 \subset R_2$ if and only if $J^l > J^h$.*

Proof See appendix.

Hence, an equilibrium fails divinity if, for some $p > v(h)$, the condition $J^l > J^h$ holds and there is some buyer who prefers to buy at p rather than at p^*, given her refined beliefs. This last condition reduces to

$$\lambda(u(h) - p) + (1 - \lambda)(u(l) - p) >$$
$$K^*(p^*)\{\sigma^*(h|p^*)[u(h) - p^*] + (1 - \sigma^*(h|p^*))[u(l) - p^*]\}. \tag{16}$$

From equation (4), $J^l > J^h$ implies:

$$\frac{\pi_s^*(l)}{p - v(l)} > \frac{\pi_s^*(h)}{p - v(h)}, \tag{17}$$

---

[11]In general, high quality sellers never deviate to any $p < v(h)$ while low quality sellers never deviate to any $p < v(l)$. This implies that, upon observing a deviation to a price $p \in (v(l), v(h))$, sellers should conclude that the deviation comes from low quality sellers.

which can be interpreted as follows. $\pi_s^*(q)/p - v(q)$ is the ratio between the opportunity cost of deviating and the gains from the deviation. If the ratio is larger for low quality sellers, sellers who deviated should be assigned a probability to be high quality at least equal to the prior λ. The following result illustrates the effect of imposing the divinity restriction on type I equilibria:

Lemma 4. *a. Given $(1 - \lambda) > \theta$, there exist FSE of type I that are divine; b. Given $(1 - \lambda) \leq \theta$, there exist divine FSE of type I if and only if $\lambda \leq \hat{\lambda} \equiv \frac{v(h) - u(l)}{u(h) - u(l)}$.*

Proof. See appendix.

As implied by sequential perfection, Divinity eliminates type I FSE where low quality sellers are the short side of the market, whenever λ exceeds the critical value $\hat{\lambda}$. By contrast, type I FSE equilibria where low quality sellers are the long side of the market are always robust to divinity. In this case, price competition forces low quality sellers to accept zero profits in equilibrium. High quality sellers experience zero profits as well since they choose not to trade. Therefore, both types of sellers face no opportunity cost of deviating, which implies that deviations have no signaling content.

The analysis of robustness of FSE of type I can be pushed one step further by invoking D1. If, for some deviation $p > v(h)$, $R_1 \subset R_2$ holds, then D1 implies that type l deviates to p with probability zero. Similarly to Divinity, an equilibrium fails D1 if there is some $p > v(h)$ at which at least one buyer is willing to buy given beliefs that assign probability zero to type l. In other words, for the equilibrium to be robust, at any possible price at which the buyer is willing to buy if he thinks the seller is high quality, low quality sellers should be at least as willing to deviate as high quality sellers. Formally, the equilibrium is robust to $D1$ if and only if

$$\frac{\pi_s^*(l)}{p - v(l)} \leq \frac{\pi_s^*(h)}{p - v(h)} \tag{18}$$

holds, for any p such that

$$u(h) - p > K^*(p^*)\{\sigma^*(h|p^*)[u(h) - p^*] + (1 - \sigma^*(h|p^*))[u(l) - p^*]\}, \tag{19}$$

i.e. for any p that makes buyers willing to deviate. Then

Lemma 5. *a. Given $(1 - \lambda) > \theta$, there exist FSE of type I that satisfy D1; b. Given $(1 - \lambda) \leq \theta$, no FSE of type I satisfies D1.*

Proof. See appendix.

D1 therefore permits to eliminate FSE of type I when low quality sellers are the short side of the market and make profits $\pi^*(l) = u(l) - v(l)$ thereby extracting all the surplus from the buyers. As we argued above, such an equilibrium does not appear

fully justifiable since low quality sellers face a large opportunity cost of deviating, whereas high quality sellers cannot be made worse off by the deviation since their equilibrium profits equal zero. It follows that, upon observing a deviation, buyers should infer that the seller deviating is high quality. Note that D1 is stronger than divinity since it requires that off-equilibrium beliefs assign probability zero to the type that has less incentive to deviate.

By contrast, when low quality sellers are the long side of the market, the opportunity cost of a hypothetical deviation is the same for low and high quality sellers. Thus, buyers cannot rule out that the deviation comes from low quality sellers and the equilibrium is therefore robust.

The following proposition summarizes the findings for type I equilibria:

Proposition 1. *a. When $1 - \lambda > \theta$ there exists a robust FSE of type I such that all trade occurs at a unique price $p^* = v(l)$ and in which only the low quality is traded. b. When $1 - \lambda \leq \theta$, any FSE of type I is not, in general, robust.*

This is implied by the analysis of lemmata 1-5 and, therefore, does not require any formal proof. So far we analysed the robustness of equilibria in which only low quality sellers are on the market. Because of the relevance of these equilibria, we explored the robustness by means of alternative criteria. We now turn our attention to whether there exist any robust equilibria in which both qualities are on the market. In this, we take a conservative approach by focusing only on D1.

5 Pooling and Hybrid equilibria

In this section we assess the robustness of pure pooling and hybrid equilibria. In doing so, we shall restrict our attention to the D1 critierion.

In any pure pooling equilibrium, by definition, there is a single equilibrium price p^* at which both high and low qualities are traded. In other words, all sellers sell at the same price p^*, which implies that they have the same probability to sell J^*. Equilibrium profits are $\pi_s^*(l) = J^*[p^* - v(l)]$ and $\pi_s^*(h) = J^*[p^* - v(h)]$ for low and high quality sellers, respectively. Condition (18) is now

$$\frac{p^* - v(l)}{p - v(l)} \leq \frac{p^* - v(h)}{p - v(h)}, \tag{20}$$

which implies

$$v(h)(p - p^*) \leq v(l)(p - p^*). \tag{21}$$

Condition (19) requires

$$u(h) - p > K^*(p^*)\{\lambda[u(h) - p^*] + (1 - \lambda)[u(l) - p^*]\} \tag{22}$$

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for $p^* \leq K^*(p^*)\{\lambda[u(h) - p^*] + (1 - \lambda)[u(l) - p^*]\}$.

Regarding hybrid equilibria, it is crucial to note that in such equilibria the same types of sellers are playing different actions. This implies that expected payoffs must be the same across the equilibrium prices, for any given seller's type. Then, we can choose the payoff associated with any price played with positive probability by type q to be type q's equilibrium payoff. Also, in any hybrid equilibrium, there is at least one price p^* at which both qualities are traded. Then, in order to assess the robustness of any hybrid equilibrium, we can focus just on the incentives to deviate from p^*. Accordingly, the equivalent of (22) for hybrid equilibria is

$$u(h) - p > K^*(p^*)\{\sigma^*(h|p^*)[u(h) - p^*] + (1 - \sigma^*(h|p^*))[u(l) - p^*]\}, \qquad (23)$$

while condition (20) stays unchanged. Note that since pooling occurs at p^*, $\sigma^*(h|p^*)$ must be strictly lower than one.

It is well known that D1 usually selects fully separating equilibria, discarding pooling and hybrid equilibria. Cho and Sobel (1990) show that in monotonic signaling games pooling may only occur at the upper bound of the action space if the action space is bounded above. Their result is however based on a sorting condition that does not always hold in this model.[12] Nevertheless, the next lemma shows that D1 retains the same effectiveness also in our set up:

Lemma 6. *No pooling/hybrid equilibrium survives D1.*

Proof. See Appendix.

In any pooling or hybrid equilibrium, sellers of type h have a higher incentive to slightly increase their prices as they make lower equilibrium profits than sellers of type l. Hence, they have again a lower opportunity cost of deviating. But buyers, aware of this, should infer that the seller who deviates is of high quality. Note also that in the case of a hybrid equilibrium (where different mixtures of high and low quality sellers are charging different prices), the deviation that unravels the equilibrium might need to be "closer" to the equilibrium price than in the pooling case, since the mix of qualities on sale at p^* does not necessarily reflect λ and $1 - \lambda$.

6 Type II separating equilibria: all types are on the market

A FSE of type II is a PBE in which low quality sellers announce a price p_l and high quality sellers announce a price $p_h > p_l$. Note that in this case, prices constitute a

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[12]The sorting condition is the following. Given p and $p' < p$, whenever $J(p,a)[p - v(l)] \geq J(p',a')[p' - v(l)]$, then $J(p,a)[p - v(h)] > J(p',a')[p' - v(h)]$. This condition is always true in our model provided that $J(p,a)$ and $J(p',a)$ are strictly positive.

perfect signal of quality: $\sigma^*(h|p_h) = 1$, and $\sigma^*(h|p_l) = 0$. Buyers obtain $u(q) - p_q^*$, with $q = l, h$, when buying at p_q. In any type II equilibrium where both qualities are traded, p_h, p_l satisfy

$$
\begin{align}
[p_l - v(l)]J^*(p_l) &\geq [p_h - v(l)]J^*(p_h) \tag{24} \\
[p_h - v(h)]J^*(p_h) &\geq [p_l - v(h)]J^*(p_l) \tag{25} \\
[u(l) - p_l]K^*(p_l) &= [u(h) - p_h]K^*(p_h) \tag{26} \\
v(l) &\leq p_l \leq u(l) \tag{27} \\
v(h) &\leq p_h \leq u(h), \tag{28}
\end{align}
$$

where $J^*(p_q) = J(p_q, \alpha^*)$ and $K^*(p_q) = K(p_q, \alpha^{1*}|a^0)$. The first and second inequalities are the Incentive Compatibility Condition (ICC) for low and high quality sellers, respectively. The third equality is the buyers' ICC. Finally, the last two inequalities refer to the participation constraints of buyers and sellers.

It can be easily verified that

Lemma 7. *If $1 - \lambda > \theta$ there is no FSE where both qualities are traded.*

This lemma does not require any formal proof once we note that in any equilibrium, low quality sellers must make zero profits whenever they outnumber the buyers. If they were not, they would undercut each other in order to increase their chance to sell their commodity. Note that undercutting would induce buyers to buy since the worst belief that buyers can assign to a seller announcing a price slightly lower than p_l is that he is type l with probability 1 and buyers were buying quality l (with probability 1) at p_l. In any equilibrium where the high quality is traded, $p_h \geq v(h)$ holds. Announcing such a price, would then ensure a strictly positive profit to low quality sellers. Therefore, the ICC of low quality sellers could not be satisfied.

In the light of lemma 7, we assume that $1 - \lambda \leq \theta$ in our quest for robust separating equilibria. Again, in order to check robustness we focus on D1.

For $p_h > p_l$, the equilibrium incentive compatibility condition of low quality sellers (ICC$_l$), given by equation (24), dictates $J^*(p_l) > J^*(p_h)$. This implies $J^*(p_h) < 1$. An undercutting argument can be used to argue that $J^*(p_l) = 1$. If it were not, low quality sellers would engage in price competition to improve their chances to sell. As we already noted, undercutting works among the low types since buyers' beliefs would never assign to any seller a quality lower than l. Accordingly, given the equilibrium payoffs associated with any fully separating equilibrium, condition (18) becomes:

$$
\frac{p_l - v(l)}{p - v(l)} \leq \frac{J^*(p_h)[p_h - v(h)]}{p - v(h)}. \tag{29}
$$

From condition (29) and ICC$_l$, it follows that a high quality seller would always signal himself by deviating to any $p > p_h$. Therefore, a necessary condition for the equilibrium

to be robust is that $K^*(p_h)$ equals 1 whenever $p_h < u(h)$. If $K^*(p_h)$ were less than 1, there would always exist a deviation $p = p_h + \epsilon$, with $\epsilon > 0$ small enough, such that a buyer would be willing to buy at p since she would get the commodity with probability one.

Hence, condition (19) becomes

$$u(h) - p > u(h) - p_h. \tag{30}$$

Note that, when $K^*(p_h) = 1$, buyers may be interested in buying only if someone deviates and announces a price p lower than p_h. In fact, in separating equilibria buyers already assign probability one to be a high quality seller to any seller posting p_h.

Condition (29) can be rewritten as

$$p \leq v(h) \frac{p_l - v(l) - \frac{v(l)}{v(h)} J^*(p_h)(p_h - v(h))}{p_l - v(l) - J^*(p_h)(p_h - v(h))}. \tag{31}$$

Note that the denominator is greater than zero from ICC_l and that the term multiplying $v(h)$ on the RHS is greater than 1 for $p_h > v(h)$.

Since the equilibrium is robust when condition (31) holds for any $p < p_h$, we can replace p by p_h without any loss of generality:

$$p_h \leq v(h) \frac{p_l - v(l) - \frac{v(l)}{v(h)} J^*(p_h)(p_h - v(h))}{p_l - v(l) - J^*(p_h)(p_h - v(h))}. \tag{32}$$

In any robust FSE p_h and p_l must satisfy the above condition. Note that $p_h \to v(h)$ as $J^*(p_h) \to 0$. This means that as the demand for high quality falls, the price should tend to the sellers' reservation price. This is not generally implied by the concept of PBE because off-equilibrium beliefs may deter sellers from lowering their prices in order to improve the chances to sell. The competitive behaviour among the high types comes back to life when off-equilibrium beliefs are chosen to satisfy D1. On the other hand, D1 does not imply that whenever $J^*(p_h) < 1$ high quality sellers should announce $v(h)$. As we argue below, in equilibrium high quality sellers might make positive profits ($p_h > v(h)$) even if $J(p_h) < 1$.

Lemma 8. *Assume $\theta > 1 - \lambda$. Then: i) both qualities are traded in any robust equilibrium; ii) the ICC of low quality sellers holds with strict equality whenever $p_h > v(h)$.*

Proof. See appendix.

The fact that the low quality is traded should not surprise. More interesting is the fact that the the high quality is traded too. The argument given in the proof goes along the following lines. As long as buyers experience positive surplus in equilibrium,

all buyers should be on the market. However, if $\theta > 1 - \lambda$, the low quality sellers are not enough to satisfy all the demand. Thus, they will raise their prices until some of the buyers will decide to buy from the high quality sellers. On the other hand, if buyers do not make any surplus ($p_l = u(l)$), given that the ICC_l must hold with equality, low quality sellers are indifferent between charging p_h and charging p_l. But then, since charging p_l gives positive profits, the probability to sell at p_h must also be positive, which implies that quality h is always traded.

The intuition of why the ICC_l must hold with equality is subtle. We have noted that in fully separating equilibria the only possible way to gain from a deviation is to announce a price slightly lower than p_h, hoping to be able to sell with a probability higher than $J^*(p_h)$. High quality sellers are willing to deviate whenever the gains from the increase in the probability to sell outweights the loss due to the small reduction in the price they charge. If the ICC_l does not hold with equality, low quality sellers strictly prefer p_l to p_h. Therefore, they are not necessarily willing to deviate if the gains for the increase in probability just outweight the loss from the reduction in price. Because of that, reasonable beliefs require that the deviation comes from the high quality. By contrast, when the ICC_l holds with equality, low quality sellers are indifferent between p_l and p_h. Therefore, they are willing to deviate whenever the high quality are, which makes the equilibrium robust.

An alternative way to interpret the result is that price competition among the high types is limited by buyers' fear that low types may step in if the probability to sell at p_h becomes too large. Hence, price competition among sellers of type h stops when the demand at p_h is such that low quality sellers are indifferent between announcing p_h and announcing p_l. Limited price competition causes high quality sellers' profits to remain positive, even when $J^*(p_h) < 1$.

In what follows, two parameters will be crucial. Define

$$\delta \equiv \frac{GFT_l}{u(h) - v(l)} \in (0, 1) \tag{33}$$

and

$$\gamma \equiv \frac{\Delta GFT}{v(h) - v(l)}, \tag{34}$$

where, for one unit of quality q, $GFT_q \equiv u(q) - v(q)$ measures the gains from trade, and $\Delta GFT = GFT_h - GFT_l$. Note that $\gamma < \delta$ holds. δ represents the overall gains from trading the low quality scaled by the range of feasible prices $u(h) - v(l)$, while γ is the difference in the gains from trade between the two qualities, ΔGFT, over the difference in the seller's evaluation of the two qualities $v(h) - v(l)$. $\gamma > 0$ implies that the gains from trading the low quality are higher than those from trading the high quality.

Lemma 9. *When $\theta \geq 1 - \lambda$, there exist robust FSE of type II. The unique outcome is: i) $J^*(p_h) = \min\left[\frac{\theta - (1-\lambda)}{\lambda}, \delta\right]$, ii) prices announced by type h and type l sellers are:*

- $p_h = u(h),\ p_l = u(l)$ *if* $\theta \geq 1 - \lambda + \lambda\delta$;

- $p_h = v(l) + \frac{\lambda}{1-\theta}[u(h) - u(l)]$ *and* $p_l = v(l) + \frac{\theta - (1-\lambda)}{1-\theta}[u(h) - u(l)]$ *if* $1 - \lambda + \gamma\lambda I_{\{\gamma>0\}} < \theta < 1 - \lambda + \lambda\delta$;

- $p_h = v(h),\ p_l = u(l) - [u(h) - v(h)]$ *if* $1 - \lambda \leq \theta \leq 1 - \lambda + \lambda\gamma I_{\{\gamma>0\}}$;

where $I_{\{\gamma>0\}} : \mathbb{R} \to \{0, 1\}$ is the indicator function.

Proof. See appendix.

Lemma 9 implies that equilibrium prices are unique and crucially depend on the buyers to sellers ratio, θ. If θ is large, trade occurs at the buyers' reservation prices: $u(h)$ and $u(l)$. If θ is small, trade occurs at prices that leave positive surplus to the buyers. For θ large/small we mean that θ is greater/smaller than the maximum fraction of total buyers who can trade without violating the ICC$-l$, which is $1 - \lambda + \lambda\delta$. Note that in this case $\theta - (1 - \lambda)$ is the number of buyers who do not manage to buy the low quality. When $\theta - (1 - \lambda)$ is larger than $\lambda\delta$ there are buyers who do not manage to buy either quality. At the same time, high quality sellers sell at $u(h)$ with a probability to sell $\delta < 1$. Otherwise, the ICC of low quality sellers would be violated. This is the adverse effect due to asymmetric information in this model: there may be buyers and sellers who do not trade even if trade would be mutually beneficial. However, D1, by requiring that the ICC$_l$ holds with equality, selects the equilibrium where the amount of trade is maximised. Thus, it picks the equilibrium in which the inefficiency is minimised, among all the possible FSE.

If $\gamma\lambda < \theta - (1 - \lambda) < \lambda\delta$ or $\gamma \leq 0$, prices are given by a markup over the lowest possible price $v(l)$. In this case, it can be shown (see the appendix) that the probability to buy at p_l must be equal to 1. This implies that all buyers who do not buy at p_l are buying at p_h. $1 - \lambda + \lambda\delta - \theta$ represents the total fraction of sellers who could potentially sell their commodities without violating the ICC (if the number of buyers were large enough) minus the actual number of buyers over the total number of sellers.

When $\theta - (1 - \lambda) \leq \gamma\lambda$ and $\gamma > 0$, such that the low quality gives higher gains than the high quality, the ICC of low quality sellers cannot hold with equality. Thus, the only possible price high quality sellers can announce is $v(h)$. This becomes rather intuitive once we note that $\gamma > 0$ implies that the gains from trading the high quality are small if compared with the gains from the low quality. Thus, in order to trade, high quality sellers should give up their profits: $p_h = v(h)$. Low quality sellers will then announce the highest possible price at which buyers are willing to buy the low quality, given the option to buy the high quality at $v(h)$.

The overall behaviour of the market prices is discontinuous in θ. This is illustrated in Figure 1.

Finally, note that lemma 9 gives a strong economic rationale for the use of D1 as equilibrium concept. Since it requires that the amount of high quality traded is the highest possible and imposes $p_h = u(h)$ whenever there are ex-post unsatisfied buyers, it implies that there is no ex-post incentive for unsatisfied buyers to buy from the high quality sellers who did not manage to sell their commodities.

7 When is the high quality harder to trade? Answer based on D1

From the analysis of type II equilibria we know that when $\theta \geq 1 - \lambda$, the only robust equilibria are type II and give a unique outcome in terms of prices and quantities. When $\theta < 1 - \lambda$, a robust equilibrium of type I, where high quality sellers stay out of the market, exists and is robust. However, this does not ensure that the equilibrium outcome is unique in this case. The next lemma shows that the property of a unique outcome holds for all D1-robust equilibria.

Lemma 10. *Given the values of the parameters θ, λ, $u(l)$, $v(l)$, $u(h)$, $v(h)$, all the resulting D1-robust equilibria display the same outcome in terms of traded qualities, quantities, and prices at which trade occurs.*

Proof See appendix.

The intuition is that when $\theta < 1 - \lambda$ the high quality is never traded. Thus, whether high quality sellers stay on the market and announce prices at which no trade occurs or stay out of the market does not affect the traded quantity and the price at which trade takes place.

This uniqueness property permits to analyse the behaviour of the traded quantities as market conditions change. The following result describes how the amount of high quality traded varies with θ:

Proposition 2. *In any D1-robust equilibrium: i. The fraction of quality l commodities being traded (over the total supply of quality l) is $\min[\theta/(1 - \lambda), 1]$; ii. The fraction of quality h commodities being traded (over the total supply of quality h) is $\max\left[0, \min\left[\frac{\theta - (1-\lambda)}{\lambda}, \delta\right]\right]$.*

The proposition is implied by the combined analysis of lemmata 5-10 and the law of large numbers. Thus, it does not require any proof.

The result is intuitive. If low quality sellers outnumber buyers ($\theta < 1 - \lambda$), price competition forces them to sell their commodities at the lowest feasible price. This effect accentuates the effect of adverse selection, making the high quality impossible to

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trade. In this case, whether high quality sellers announce a price or not is irrelevant. By converse, if low quality sellers are less than the buyers, price competition at the low price is less fierce. Also, in this case, robustness requires that high quality sellers should never remain out of the market (i.e. they always announce a price). This ensures that the high quality is traded in equilibrium, though, for high quality sellers, the probability to make a sale is less than one.

Inspection of the expression in proposition 2, shows a discontinuous relationship between the relative demand, θ, and the percentage of high quality sold. If $\theta \leq 1 - \lambda$, the high quality is never traded. If $\theta > 1 - \lambda$ the amount of high quality trade linearly increases in θ. However, the percentage of high quality traded cannot grow too much as to violate the incentive compatibility of low quality sellers. Hence, when $\theta - (1 - \lambda) \geq \lambda\delta$, the amount of high quality being traded must remain constant. This is depicted in figure 2.

A close look at the parameter $\delta = \frac{u(l) - v(l)}{u(h) - v(l)}$, reveals the presence of a "superior-quality's curse". Since δ is decreasing in $u(h)$, if the high quality is extremely appreciated by buyers, the probability to sell it must be very low. The intuition is that if $u(h)$ is large, the price of the high quality should be very high. On the other hand, separation requires that the incentive compatibility of low quality sellers is satisfied. Thus, the probability to sell the high quality must be low.

To conclude the section, we give a brief account of what happens in the case the assumption $v(h) > u(l)$ is violated. The reader should note that the analysis of robust equilibria is exactly the same. What changes are the characteristics of some non-robust equilibria. For instance, in the equilibrium of type I where $\theta > 1 - \lambda$, it is clear that low quality sellers would always announce a price slightly below $v(h)$ instead of announcing $u(l)$. The existence of some pooling and hybrid equilibria is also affected.

8 Implications for efficiency

In this section we analyze the welfare implications of the model with strategic price-setting, by comparison with the perfect competition case, where buyers are fully informed, which provides the first best benchmark. In doing so, we shall concentrate on the equilibria robust to the D1 criterion. As an additional term of comparison, we shall also refer to the equilibrium outcome(s) of a Walrasian model of centralized market, where trade occurs at a single price set by an auctioneer.

Under perfect competition, both the quantities and the qualities exchanged are always those which maximise the ex post gains from trade. Depending on whether $\Delta GFT > 0$ holds or not, the corresponding level of welfare being generated through

the exchange when markets are perfect is

$$W_{FI} = \begin{cases} GFT_l \min(B,S) + \Delta GFT \min(B, \lambda S) & \text{if } \Delta GFT > 0 \\ GFT_l \min(B,S) + \Delta GFT \max[\min(B,S) - (1-\lambda)]S, 0] & \text{if } \Delta GFT < 0. \end{cases}$$

(35)

Based on proposition 2, the welfare level generated by the model with strategic price setting and asymmetric information can be written as

$$W = \begin{cases} GFT_l B & \text{if } B \in [0, (1-\lambda)S) \\ GFT_l B + \Delta GFT(B - 1 - \lambda)S & \text{if } B \in [S(1-\lambda), S(1-\lambda+\delta\lambda)] \\ GFT_l S(1-\lambda+\delta\lambda) + \delta\lambda S \Delta GFT & \text{if } B \in (S(1-\lambda+\delta\lambda), \infty). \end{cases}$$

(36)

We compare (36) and (35) in each of the following two distinct cases:

a. Gains from trading quality l exceed gains from trading quality h: $\Delta GFT < 0$;

b. Gains from trading quality h exceed gains from trading quality l: $\Delta GFT > 0$.

Let us focus on case (a), first. On the basis of (35) and (36), the relative welfare loss associated with the model where sellers announce the prices is

$$\Delta w_{FI} = 1 - \frac{W}{W_{FI}} = \begin{cases} 0 & \text{if } \theta \leq 1 - \lambda + \delta\lambda \\ 1 - \frac{GFT_l(1-\lambda+\delta\lambda) + \lambda\delta\Delta GFT}{\min(\theta,1)GFT_l + \Delta GFT \max(\min(\theta,1) - (1-\lambda),0)} & \text{if } \theta > 1 - \lambda + \delta\lambda. \end{cases}$$

(37)

According to this expression, the market with strategic price-setting achieves the first best level of welfare as long as $\theta \leq (1 - \lambda + \delta\lambda)$. Viceversa, if $\theta > (1 - \lambda + \delta\lambda)$, an inefficiency emerges due to the fact that there is too little trade. D1 selects the equilibria in which the amount of trade is maximised given the ICC of low quality sellers. As a consequence, the market suffers an efficiency loss if and only if the demand is sufficiently large. The behaviour of Δw_{FI} is described in figure 3, which shows how Δw_{FI} increases with θ.

It is worth noting that, under the same circumstances, the centralized Walrasian market where trade occurs at a single price could potentially lead to a different outcome. As long as $\theta < (1 - \lambda)$ this model would imply a unique equilibrium where only quality l would be traded and, again, that outcome would be efficient. However, whenever $\theta > (1 - \lambda)$, we have to distinguish two possible situations depending on whether $\lambda \geq \hat{\lambda}$ holds or not, where $\hat{\lambda}$ has been defined in lemma 2. If this inequality is not satisfied, then the Walrasian model leads a unique equilibrium where only quality l is exchanged, and the amount of trade equals $S(1 - \lambda)$. This outcome would be inefficient as long as $\theta > (1 - \lambda)$ holds.

On the other hand, if $\lambda \geq \hat{\lambda}$, the hypothesis of price-taking behaviour leads to two possible equilibria: (i) a separating equilibrium where only quality l is traded, in the

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amount $S(1-\lambda)$; (ii) a pooling equilibrium where both qualities are traded. Here, the amount of trade would be $\min(B,S)$, of which a share $(1-\lambda)$ would be of quality l and λ would be of quality h. Clearly enough, under the separating equilibrium the welfare loss implied by the Walrasian model is still greater than the welfare loss, if any, associated with the strategic-price setting model.

Things are different under the pooling equilibrium. The level of welfare of the Walrasian model in this case is

$$W_W = GFT_l \min(B,S) + \lambda \Delta GFT[\min(B,S)]. \tag{38}$$

Let Δw_W denote the welfare loss implied by the strategic price setting model relatively to the Walrasian model. Taking into account that for $\theta \le 1-\lambda$ the Walrasian model and the model with strategic price setting share the same outcome, and comparing (38) and (36) yields

$$\Delta w_W = 1 - \frac{W}{W_W} = \begin{cases} 0 & \text{if } \theta \in [0, 1-\lambda] \\ 1 - \frac{GFT_l + \Delta GFT(1-(1-\lambda)/\theta)}{GFT_l + \lambda \Delta GFT} & \text{if } \theta \in (1-\lambda, 1-\lambda+\delta\lambda) \\ 1 - \frac{GFT_l(1-\lambda+\delta\lambda)+\lambda\delta\Delta GFT}{\min(\theta,1)GFT_l + \lambda\Delta GFT \min(\theta,1)} & \text{if } \theta \in [1-\lambda+\delta\lambda, \infty). \end{cases} \tag{39}$$

As depicted in figure 4, Δw_W is zero for $\theta \in [0, 1-\lambda]$, negative for θ less or equal to $\hat{\theta}$, and positive otherwise, where

$$\hat{\theta} = \frac{GFT_l(1-\lambda+\lambda\delta) + \delta\lambda\Delta GFT}{GFT_l + \lambda\Delta GFT}. \tag{40}$$

It can be easily verified that $\hat{\theta} \in (1-\lambda+\delta\lambda, 1)$.

Assume the Walrasian auctioneer is able to select the pooling equilibrium equilibrium, so that trade is maximised. Then, as the value of θ increases, the Walrasian market will eventually outperform the decentralized market where sellers are price-setting. The intuition behind this result is that, so long as pooling equilibria are selected, the Walrasian market, differently from the other model, provides the efficient amount of trade for any value of $\theta > (1-\lambda)$. Certainly, the quality of trade could still be different from that of the first best case. However, from the point of view of efficiency, the quality aspect of trade becomes less of an issue compared to quantity as θ increases. This is why, for θ sufficiently high, the Walrasian model outperforms the model with strategic price setting.

Let us now turn to the case (b), in which the gains from trading quality l are less than the gains from trading quality h. In this case, the model with strategic price-setting never yields an efficient outcome. The relative welfare loss with respect to the

case of perfect markets is

$$\Delta w_{FI} = \begin{cases} 1 - \frac{GFT_l\theta}{GFT_l\theta + \Delta GFT \min(\theta,\lambda)} & \text{if } \theta \in [0, 1 - \lambda] \\ 1 - \frac{GFT_l\theta + \Delta GFT[\theta - (1-\lambda)]}{\theta GFT_l + \Delta GFT \min(\theta,\lambda))} & \text{if } \theta \in (1 - \lambda, 1 - \lambda + \delta\lambda) \\ 1 - \frac{GFT_l(1 - \lambda + \delta\lambda) + \delta\lambda\Delta GFT}{GFT_l \min(\theta,1) + \min(\theta,\lambda)\Delta GFT} & \text{if } \theta \in [1 - \lambda + \delta\lambda, \infty). \end{cases} \quad (41)$$

The behaviour of w_{FI} is described in figure 5, where the w_{FI} has been drawn under the hypothesis that $\lambda < 0.5$ holds (the graph does not change significantly if the other case is considered). As figure 5 shows, strategic price setting always results in a welfare loss. As θ increases and some of the h quality starts to be traded, the loss decreases. But then, as θ reaches the maximum percentage of the overall supply which can be traded under strategic price setting, $1 - \lambda + \delta\lambda$, the relative loss shoots up again to reach its maximum value for $\theta = 1$.

What is the welfare loss associated with the Walrasian model under case (b)? The relative welfare loss of our model compared to the Walrasian centralized market is still given by expression (39), where ΔGFT is now greater than zero. The behaviour of Δw_w is described in figure 6. For any $\theta > 1 - \lambda$ the model with strategic price setting gives a lower welfare than the Walrasian model. This, provided that a pooling equilibrium exists and the Walrasian auctioneer selects it. The loss decreases for values of θ between $1 - \lambda$ and $1 - \lambda + \delta\lambda$ as quality h starts to be traded, and increases again for values of θ greater than $1 - \lambda + \delta\lambda$ due to the high quality's curse. Assume that $\lambda \geq \hat{\lambda}$ and the Walrasian auctioneer always selects the equilibrium in which trade is maximised. Then, for $\theta \geq 1$, the Walrasian market yields the first best level of welfare. By contrast, in the market with strategic pricing, too little trade occurs and an excessively high proportion of quality l is exchanged.

Summing up, if the demand is less than the supply of low quality, both the Walrasian and the strategic price setting models yield the same result. Otherwise, the welfare implications of the two models can be very different. If the pooling equilibrium which maximises trade in the Walrasian model does not exist, then strategic price-setting generally yields a superior outcome in terms of welfare. If, on the other hand, the pooling equilibrium exists, then the Walrasian model could do better. This certainly happens if $\Delta GFT > 0$ and might happen when $\Delta GFT < 0$ but only if demand is strong enough to trigger the inefficiency caused by the high quality's curse that plagues the model with strategic pricing.

9   Extension to any finite number of qualities

This section generalises the results concerning the set of equilibria robust to D1 to the case of a finite number $(N + 1)$ of qualities. Qualities are indexed by $q = 0, ..., N$. Each seller's quality is drawn from a distribution $\lambda : \{0, 1, ..., N\} \rightarrow [0, 1]$, where λ_q,

$\sum_{q=0}^{N} \lambda_q = 1$, denotes the probability associated with quality q. Buyers' posterior beliefs are denoted by $\sigma(q|a_s^0)$, $\sum_{q=0}^{N} \sigma(q|a_s^0) = 1$.

Let us concentrate first on pooling and hybrid equilibria in which two or more types of sellers announce the same price. In order to assess the robustness of these equilibria, we need to understand how the equivalents of conditions (20) and (22) look like. In any of these equilibria there always exists a price p^* at which a non-singleton non-empty set of qualities, $M \subseteq \{0, ..., N\}$, is traded. Let q_M be the highest quality in M. Take any quality $q \in M$, $q \neq q_M$. Condition (20) becomes:

$$\frac{p^* - v(q)}{p - v(q)} \leq \frac{p^* - v(q_M)}{p - v(q_M)}, \tag{42}$$

which, given $v(q) < v(q_M)$, is violated for any $p > p^*$. Thus, reasonable beliefs should assign probability 0 to a deviation $p > p^*$ by any quality in M except for q_M. As for qualities $q \notin M$, the following applies. Sellers of qualities $q < q_M$ who do not trade at p^* do so because they make higher profits charging a different price. Therefore, they should be assigned probability zero. Sellers of qualities $q > q_M$ should also be assigned probability zero so long as $p < v(q_M + 1)$. Thus, deviations $p^* < p < v(q_M + 1)$ are attributed to sellers of quality M with probability 1. As for buyers, the equivalent of condition (22) is:

$$u(q_M) - p > K^*(p^*) \sum_{q \in M} \sigma^*(q|p^*)[u(q) - p^*], \tag{43}$$

which is always true for p close enough to p^*. Thus, neither pooling nor hybrid equilibria in which two or more types announce the same price survive D1.

The set of robust equilibria therefore includes only fully separating equilibria and partially separating equilibria with the necessary condition that sellers on the market fully separate. We now focus on these equilibria. If $\theta \leq \lambda_0$, the discussion made in the previous sections leads to the immediate conclusion that only quality 0 is traded and the only price in the market is $p_0 = v(0)$. When $\theta > \lambda_0$, in general, more than one quality is traded. In this case, the relevant incentive compatibility condition for sellers is:

$$J(p_{q-s})[p_{q-s} - v(q-s)] \geq J(p_q)[p_q - v(q-s)] \tag{44}$$

for all q and $s = 0, ...q$. The usual undercutting argument implies that $J(p_0) = 1$ holds. This, together with (44) yields $p_q > p_{q-1} > ... > p_0$ and $J(p_q) < J(p_{q-1}) < ... < J(p_1) < 1$. From equation (44):

$$J(p_{q-1})[p_{q-1} - v(q-1)] \geq J(p_q)[p_q - v(q-1)] \tag{45}$$

and

$$J(p_q)[p_q - v(q)] \geq J(p_{q+1})[p_{q+1} - v(q)]. \tag{46}$$

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Then, by using $v(q) > v(q-1)$ and $J(p_q) > J(p_{q+1})$, it follows that

$$J(p_{q-1})[p_{q-1} - v(q-1)] \geq J(p_{q+1})[p_{q+1} - v(q-1)] \tag{47}$$

always holds. Therefore, when the "adjacent upward" ICC is satisfied, all the ICC with respect to all higher qualities must be satisfied.

As explained in section (6), in any robust equilibrium in which buyers make a positive surplus, $K^*(p_q)$ must be 1 for any quality q that is traded in equilibrium. Then, Buyers' ICC is:

$$u(q) - p_q = k \; \forall q = 0, 1, ..., N, \tag{48}$$

where k is buyers' net payoff from buying any of the qualities.

For $q > 0$, buyers may only be attracted by deviations $p < p_q$, since they are already able to buy quality q with probability 1 at p_q. This implies that only $p < p_q$ are relevant deviations. D1 implies that type q should be assigned probability zero of deviating to price p, if there exists q' such that:

$$\frac{J(p_q)[p_q - v(q)]}{p - v(q)} > \frac{J(p_{q'})[p_{q'} - v(q')]}{p - v(q')} \tag{49}$$

This is the equivalent of condition (29).

Lemma 11. *For all $q = 1, ..., N$, in any robust equilibrium in which quality q is traded, the "adjacent upward" ICC of sellers of quality $q-1$ holds with equality whenever $p_q > v(q)$.*

Proof See appendix.

We now characterise the robust equilibria. Assume that all sellers announce a price $p_q > v(q)$ and all qualities are traded. We rewrite the sellers' ICCs to obtain

$$J(p_q) = J(p_{q-1}) \frac{u(q-1) - v(q-1) - k}{u(q) - v(q-1) - k}. \tag{50}$$

Using the initial condition $J(p_0) = 1$, we solve equation (50) to get:

$$J(p_q) = \prod_{i=0}^{q-1} \frac{u(q-i-1) - v(q-i-1) - k}{u(q-i) - v(q-i-1) - k}. \tag{51}$$

Since $k \geq 0$, the maximum value of $J(p_q)$ is achieved when $k = 0$. Defining

$$\delta_q \equiv \frac{u(q-1) - v(q-1)}{u(q) - v(q-1)}. \tag{52}$$

the probability to trade at p_q is bounded above by:

$$\overline{J}_q = \prod_{i=0}^{q-1} \delta_{q-i}. \tag{53}$$

Note also that $k = 0$ implies $p_q = u(q) \ \forall q = 0, ..., N$. Therefore, buyers make zero surplus. Assume $k > 0$, so that buyers make positive surplus at every price. Then, all buyers must stay on the market. This implies:

$$\sum_{q=0}^{N} \lambda_q J(p_q) = \theta, \tag{54}$$

where $J(p_q)$ is given by (51). In this case, an equilibrium is a value k^* solving (54). Inspection of expression (54) shows that the LHS is monotonically decreasing in k, for $0 \leq k \leq u(q) - v(q) \ \forall q = 0, ..., N$. Therefore, there always exists at most one admissible k^* in the above interval, which is also the relevant interval since it takes into account sellers' and buyers' participation constraint. This also implies that if there exists an equilibrium, it must be unique. Regarding the existence, consider the following. The LHS of expression (54) reaches its maximum in the relevant interval when $k = 0$ and its its minimum when $k = \hat{k} \equiv \min_{q \in \{o,...,N\}} u(q) - v(q)$. Therefore, necessary and sufficient conditions for an equilibrium where $0 \leq k^* \leq u(q) - v(q), \ \forall q$ to exist are

$$\theta \leq \sum_{q=0}^{N} \lambda_q \prod_{i=0}^{q-1} \delta_{q-i} \tag{55}$$

and

$$\theta \geq \sum_{q=0}^{N} \lambda_q \prod_{i=0}^{q-1} \frac{u(q-i-1) - v(q-i-1) - \hat{k}}{u(q-i) - v(q-i-1) - \hat{k}}. \tag{56}$$

Otherwise, if one of the above conditions is not satisfied, the robust equilibrium takes a different form. If condition (55) is not satisfied, k is equal to zero and the equilibrium is characterized by $p_q = u(q)$ for all $q = 0, 1, ..., N$ and probabilities $\overline{J}_q$ given by (53).

Assume now that condition (56) is not satisfied. Let $\underline{J}_q$ be the probability to sell at p_q when k equals $\hat{k}$, and note $k \leq \hat{k}$ must hold. If the value of the RHS of equation (56) for $k = \hat{k}$ is greater than θ, then some qualities are not traded. To see this, note that $\underline{J}_q$ is zero for all $q > \hat{q}$, where $\hat{q} = \arg\min_{q \in \{0,...,N\}} u(q) - v(q)$. [13] Thus, when $k = \hat{k}$, all qualities above the quality which provides the lowest gain from trade are

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[13]If the minimum is not unique, then $\underline{J}_q$ equals zero for all qualities above the lowest quality in $\arg\min_{q \in \{0,...,N\}} u(q) - v(q)$.

not traded. Buyers' surplus k should increase, however it fails to increase because, by increasing, it would violate the participation constraint of sellers of type $\hat{q}$. Hence, the incentive compatibility for type $\hat{q}$ requires that no higher quality is traded.

What are the characteristics of the equilibrium in this case? Note that if (56) is not satisfied we have:

$$\sum_{q=0}^{\hat{q}} \lambda_q \underline{J}_q = \sum_{q=0}^{N} \lambda_q \underline{J}_q > \theta, \tag{57}$$

where the first equality comes from the fact that all qualities higher than $\hat{q}$ are not traded. This suggests that even if only $\hat{q} + 1$ qualities are traded out of $N + 1$, the quantity sold is still higher than the quantity bought. Then, quality $\hat{q}$ cannot be traded in equilibrium. Let us assume that quality $\hat{q} - 1$ is traded. It follows that its price, $p_{\hat{q}-1}$, must be compatible with D1. In other words, there must be no incentive to deviate for sellers of type $\hat{q}$ or above. Therefore, $p_{\hat{q}-1}$ must be such that $k_1 \equiv u(\hat{q} - 1) - p_{\hat{q}-1} \geq u(\hat{q}) - v(\hat{q})$. If so, there is no price sellers of type $\hat{q}$ could possibly announce to attract buyers and still make no loss. Of course, buyers incentive compatibility implies $u(q) - p_q = k_1$ for all qualities that are traded, i.e. $q = 0, ..., \hat{q} - 1$. Let now $\tilde{q} \equiv \arg\min_{q \in \{0,...,\hat{q}-1\}} u(q) - v(q)$ denote the quality that gives the lowest gain from trade among the traded qualities. It is clear that k_1 should now satisfy $u(\hat{q}) - v(\hat{q}) \leq k_1 \leq u(\tilde{q}) - v(\tilde{q})$. Therefore, all that remains to be checked is whether there exists k_1^* such that:

$$\sum_{q=0}^{\hat{q}-1} \lambda_q \prod_{i=0}^{q-1} \frac{u(q-i-1) - v(q-i-1) - k_1^*}{u(q-i) - v(q-i-1) - k_1^*} = \theta. \tag{58}$$

If it does, then the equilibrium is such that qualities $q = 0, ..., \hat{q} - 1$ are traded. If it does not, then all the process starts again by choosing $\tilde{q}$ as the first quality that is not traded. It should be noted that, since we are assuming $\theta > \lambda_0$, the process eventually leads to an equilibrium in which more than one quality is traded. In fact, as long as $\theta > \lambda_0$, qualities 0 and 1 are always traded. Interestingly enough, if gains from trade are non-decreasing in q, then all quality are necessarily traded in equilibrium. This happens because the quality which displays the minimum gains is quality 0. Then, k must be lower or equal to $u(0) - v(0)$. As k reaches $u(0) - v(0)$, the LHS in equation (54) goes to λ_0 since all $J(p_q)$ are zero except $J(p_0)$ which is equal to 1. Therefore, as long as $\lambda_0 < \theta$, an internal solution in which all qualities are traded exists.

The extension to $N + 1$ qualities enables us to understand the special role that the lowest quality has in our model. If $\theta \leq \lambda_0$, then only the lowest quality is traded. Otherwise, more than one quality is always traded and, possibly, all $N + 1$ qualities are traded. The lowest quality is peculiar because its trade is not plagued by information imperfection and price competition works perfectly. This does not necessarily hold for higher qualities.

10 Conclusions

This paper tackled the issue of strategic pricing in a competitive market for lemons where the potential gains from trade are positive. The seller's choice of announcing a given price is affected by two types of considerations. The first is the desire to sell at a price that ensures the highest net profit and, at the same time, maximises the chance to find a buyer. The second is the willingness to conceal/reveal his true quality through the price he announces.

By imposing standard restrictions on buyers' beliefs upon observing a deviation to a price that is never announced in equilibrium, we are able to fully charaterise the set of robust equilibria. This is composed of fully separating equilibria in which different types of sellers choose different actions. In these equilibria, the low quality is always traded. Market conditions (the relative number of buyers to sellers) are crucial in determining whether the best quality will be traded and in what amount. As the relative demand increases, the two qualities are traded in sequence. In particular, if low quality sellers are able to satisfy the whole demand, the high quality is never traded. The idea is that competition among the low quality sellers becomes so fierce that their profits are driven to zero. Under these circumstances, if the high quality were traded, low quality sellers would always profit from announcing the price at which the high quality is sold. Hence, incentive compatibility requires that high quality sellers never trade. This result contravenes conventional wisdom in that it is independent of the buyers' and sellers' relative evaluations of the two qualities. The high quality is never traded even if the difference between the buyer and the seller reservation prices is arbitrarily large.

If low quality sellers are not enough to satisfy the whole demand, then the high quality is always traded, albeit the probability to make a sale for the individual sellers is strictly less than 1. The reason is that the high quality can be only traded at relatively high prices. Then, incentive compatibility for low quality sellers requires that the chance to make a sale at those prices is small. As the demand grows, the fraction of high quality goods exchanged increases until a threshold is reached and remains thereafter constant. This implies that even if buyers outnumber sellers, there are always sellers who are unable to sell their commodity.

When the demand is strong, we identify the presence of a quality paradox. The more the best quality is appreciated by the potential buyers, the lower the amount that is traded in equilibrium. The paper is also interesting from a technical viewpoint. We consider strategic signaling in a competitive market characterised by a large number of agents. Hence, our setup is ideal to study how agents' competitive behaviour may be limited by information imperfections. This is made possible by our assumption that buyers observe the prices announced by individual sellers implying that prices are signals of individual qualities rather than market-average qualities, as in early works. Finally, the paper compares alternative solution concepts to identify the most

reasonable outcome from an economic perspective.

Appendix

Proof of lemma 1

Any PBE of type I satisfies definition 2. A necessary condition for this to happen is $p^* < v(h)$ for all $p^* \in \mathbf{p}^*$ so that posting p^* is strictly dominated by $a_s^{0*} = n$ for any seller of type h. Bayes' rule then implies $\sigma^*(h|p^*, a_{-s}^{0*}) \equiv \sigma^*(h|p^*) = 0$ for any $p^* \in \mathbf{p}^*$, so that at p^* buyers' maximum willingness to pay equals $u(l)$. But then, $p^* \leq u(l)$ must hold for any equilibrium price in a PBE of type I. Moreover, given that $v(l)$ is low quality sellers' reservation price, $p^* \geq v(l)$ must hold. Hence, in any PBE of type I where $\mathbf{p}^*$ is non-empty, $u(l) \geq p^* \geq v(l)$ for all $p^* \in \mathbf{p}^*$. We prove that: a. The equilibrium is characterized by a unique price, if some price is announced; b. If low quality sellers are the short (long) side, the equilibrium characterized by $p^* = u(l)$ ($p^* = v(l)$) is the only equilibrium where prices are announced; c. No seller announcing a price is never an equilibrium.

    a. Uniqueness of price. Suppose there is more than one price. Consider for instance the case of two different equilibrium prices p' and p'' with $p'' > p'$. Clearly, p', p'' could be a pair of equilibrium prices if and only if $\pi_s(p', a_{-s}^*|l, \sigma^*) = \pi_s(p'', a_{-s}^*|l, \sigma^*)$ which implies that sellers are the long side of the market at p'' and the short side at p' so that $J(p'', \alpha^*) < J(p', \alpha^*)$. Otherwise, posting p' would be dominated by p''. So, $J(p'', \alpha^*)$ must be lower than 1. But then, any seller posting p'' would profit from lowering p'' by an infinitesimal amount in order to attract buyers buying at p''. It is important to note that this deviation is profitable for any off-equilibrium beliefs of the buyers: the quality assigned to such a deviation can never be lower than l. Moreover, given other buyers' actions, a single buyer who deviates and buys at this lower price, will be able to buy with probability 1. The same reasoning applies to any number of prices higher than one. Thus, the only possible equilibrium is when there is a single price p^*.

    b. Short-long side. If $(1 - \lambda) > \theta$, sellers of type l are the long side of the market. In this case, any $p^* > v(l)$ cannot be an equilibrium. Each individual seller would have incentive to deviate and announce $p^* - \epsilon$ where $\epsilon > 0$ is sufficiently small, as an infinitesimal reduction in the price would result in a non-infinitesimal positive change in the level of profits since to the probability of selling jumps from $\theta/(1-\lambda)$ to 1. Again, note that this deviation is profitable whatever buyers' beliefs are. Undercutting leads to $p^* = v(l)$. Clearly, undercutting makes no sense if $(1 - \lambda) \leq \theta$. In this case, each seller of type l can charge the buyers the maximum price they are willing to pay. Hence, if $(1 - \lambda) \leq \theta$, $p^* = u(l)$ must hold. Finally, for $p^* = u(l)$ and $p^* = v(l)$ to be equilibrium prices, there must be a class of off-equilibrium beliefs such that none has any incentive to announce a different price, greater than $u(l)$. It is easy to verify that such beliefs exist. For instance, beliefs assigning $\sigma(h|p, a_{-s}^{0*}) = 0$ to any seller announcing a price $p \neq p^*$ support the PBE.

    c. $\mathbf{p}^* = \emptyset$ *is never an equilibrium.* Consider an equilibrium where all sellers

are off the market, i.e. $a^{0*} = \{n, n, ..., n\}$. Buyers would be willing to buy at any price $p < u(l)$ for any possible off equilibrium beliefs. Then, entering the market and posting $p \in (v(l), u(l))$ is always profitable for a seller s of type l, since $\pi_s(p, a'_{-s}|l, \sigma^*) > \pi_s(n, a^*_{-s}|l, \sigma^*) = 0$, where a' is the action profile which includes buyers' best response to p. $\square$

Proof of lemma 2

Case a. The equilibrium price is $p^* = v(l)$. Equilibrium profits are $\pi_s^*(h) = \pi_s^*(l) = 0$. Set $Q = \{h\}$ and $\sigma(h|p) = 1$. Then, provided that buyers have any incentive to buy, all sellers would profit from deviating to any $p \geq v(h)$, while high quality sellers would never deviate to any $p < v(h)$. Thus, $\sigma(h|p) = 1$ would necessarily be disconfirmed. Set now $Q = \{l\}$ and $\sigma(l|p) = 1$. In this case, for any $p > v(l)$, buyers' expected profits would be higher at p^* than at p. Then, the best response is not to buy. Yet, nor type l neither type h are actually made worse or better off by announcing p. But then, since type l weakly benefits and type h weakly loses, $\sigma(l|p) = 1$ is credible.

Case b. The equilibrium price is $p^* = u(l)$. In equilibrium, low quality sellers make profits $\pi_s^*(l) = u(l) - v(l)$, while high quality sellers make zero profits and buyers zero surplus. Set $Q = \{h\}$, so that $\sigma(h|p) = 1$. For this to be the case, $p \geq v(h)$ must hold. Then, provided that $p < u(h)$ buyers best reply is to buy at p. But then, any of the sellers would profit from deviating so that $\sigma(h|p) = 1$ would be invalidated. Hence, $\sigma(h|p) = 1$ is not credible. Set $Q = \{l\}$ and $\sigma(l|p) = 1$. For any $p > u(l)$ the best reply is not to buy so that $\sigma(l|p) = 1$ is again disconfirmed and therefore is not credible. Finally, set $Q = \{h, l\}$. Then assume a deviation $p > v(h)$ (but close enough to $v(h)$). Then, $\Pr(Q) = 1$, $\Pr(h) = \lambda$, and $\Pr(l) = 1 - \lambda$. If $\lambda u(h) + (1 - \lambda)u(l) \geq p$, the best response is to buy following the deviation. All sellers have an incentive to deviate such that beliefs are confirmed. Thus, the equilibrium where $p^* = u(l)$ will be deviated whenever $\lambda u(h) + (1 - \lambda)u(l) > v(h)$. Solving for λ yields the threshold:

$$\hat{\lambda} \equiv \frac{v(h) - u(l)}{u(h) - u(l)}. \tag{59}$$

So if $\lambda > \hat{\lambda}$ the equilibrium outcome $p^* = u(l)$ does not satisfy the SPR. $\square$

Proof of lemma 3

Clearly, whenever $J^l > J^h$, $R_1 \subseteq R_2$ follows. Therefore, in order to prove the result it is sufficient to show that there always exists a profile α^1 which implies a probability to sell $J(p, \alpha) \in (J^h, J^l)$ and $\alpha^1 \in MBRP(a^0, \hat{\sigma})$ for some $\hat{\sigma}$. If we show that, for any $j \in [0, 1]$, there always exists a profile $\alpha^1 \in MBRP(a^0, \hat{\sigma})$ that gives $J(p, \alpha) = j$, then we have proved the result. Take any profile $\overline{\alpha}^1$ such that the buyers' individual

probabilities to play p are given by $\overline{\alpha}^1(p) = \{0, 0, ..., j, ..., 0\}$. Then $J(p, \overline{\alpha}) = j$ holds. All that remains to show is that there is always $\hat{\sigma}$ such that $\overline{\alpha}^1 \in MBRP(a^0, \hat{\sigma})$. Let b be the buyer who plays p with probability j when all other buyers play p with probability zero. The payoff of buyer b when playing $a_b^1 = p$ and beliefs are given by $\hat{\sigma}(h|p)$ is:

$$\pi_b(p) \equiv \hat{\sigma}(h|p)[u(h) - p] + (1 - \hat{\sigma}(h|p))[u(l) - p]. \tag{60}$$

The payoff when playing $a_b^1 = p^* \in \mathbf{p}$ is:

$$\pi_b(p^*) \equiv K^*(p^*)\{\sigma^*(h|p^*)[u(h) - p^*] + (1 - \sigma^*(h|p^*))[u(l) - p^*]\}, \tag{61}$$

where, as usual, we rely on point *iii* of definition 1 and on the fact that the market is large and therefore $K^*(p^*)$ does not change after the deviation of one seller/buyer. If playing p with probability $j > 0$ is a best response for b, then (60) must be equal to (61). Thus, solving for $\hat{\sigma}$:

$$\hat{\sigma}(h|p) = K^*(p^*)\sigma^*(h|p^*) + \frac{p - K^*(p^*)p^* - (1 - K^*(p^*))u(l)}{u(h) - u(l)}. \tag{62}$$

Note that non-emptiness or R_2 implies that $\hat{\sigma}(h|p) \leq 1$. Also, playing p with probability zero must be optimal for all buyers different from b. Note that if buyer b plays p with probability $j > 0$, the probability to obtain the good at p for all other buyers is $K(p) = 1 - \frac{j}{2} < 1$. Thus, equation (60) becomes:

$$\pi_r(p) = K(p)\{\hat{\sigma}(h|p)[u(h) - p] + (1 - \hat{\sigma}(h|p))[u(l) - p]\}, \; r \in \mathcal{B}, r \neq b. \tag{63}$$

Substituting $\hat{\sigma}$ from equation (62) gives:

$$\pi_r(p) = K(p)\pi_r(p^*) < \pi_r(p^*) \; \forall r \in \mathcal{B}, \; r \neq b. \tag{64}$$

Therefore, when buyer b plays p with probability j and beliefs are given by $\hat{\sigma}$, playing p with probability zero is a best response for all other buyers. □

Proof of lemma 4

Case a. Consider first the case in which $\theta > (1 - \lambda)$. Note that if this inequality holds both types make zero profits, thus condition (17) becomes $0 = 0$ and high quality sellers are never more likely to deviate than low quality. Thus, the equilibrium is robust to divinity.

Case b. Consider type *I* FSE where $\theta \leq (1 - \lambda)$ so that low quality sellers are the short side of the market. In equilibrium, high quality sellers make zero profits, $\pi_s^*(h) = 0$, while low quality sellers make profits $\pi_s^*(l) = u(l) - v(l)$. Thus, for $J(p, \alpha) > 0$ and $p > v(h)$, condition (17) becomes:

$$\frac{u(l) - v(l)}{p - v(l)} > 0, \tag{65}$$

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which implies that, for any $p > v(h)$, high quality sellers need a lower probability J^h to be willing to deviate than low quality sellers. Also, buyers make zero surplus, such that the condition (16) becomes

$$\lambda(u(h) - p) + (1 - \lambda)(u(l) - p) > 0 \tag{66}$$

for some p. Since p must be greater than $v(h)$, the necessary and sufficient condition is

$$\lambda(u(h) - v(h)) + (1 - \lambda)(u(l) - v(h)) > 0, \tag{67}$$

which implies that the equilibrium fails divinity if $\lambda > \hat{\lambda}$, where $\hat{\lambda}$ is the same as in lemma (2). □

Proof of lemma 5

Case a. It is immediate to show that type I equilibria satisfy D1 when $1 - \lambda > \theta$. Inspection of condition (18) yields $0 \geq 0$ that is always true for any p. Therefore, for any p, $J^h = J^l$. Such equilibria are always robust to D1.

Case b. Type I equilibria where $1 - \lambda \leq \theta$ always fail D1. In equilibrium, low quality sellers make profits $\pi^*(l) = u(l) - v(l)$ while high quality sellers make zero profits. Condition (18) becomes, for any $p > v(h)$:

$$\frac{u(l) - v(l)}{p - v(l)} \leq 0, \tag{68}$$

which is not met by any $p > v(h) > v(l)$, which implies $J^l > J^h$. Also, condition (19) becomes $u(h) - p > 0$ which implies that any deviation $p \in (v(h), u(h))$ would destroy the equilibrium. □

Proof of lemma 6

To begin with, we note that p^* must be always strictly lower than $u(h)$, otherwise buyers would make a loss. Condition (20) is never verified for $p > p^*$. So, for deviations above the pooling equilibrium price, high quality sellers are more likely to deviate. As for condition (22), buyers prefer to buy at p as long as p is lower than $p^* + \eta$, $\eta \equiv (1 - \lambda)K^*(p^*)[u(h) - u(l)] + (1 - K^*(p^*))u(h) > 0$. Thus, any deviation $p \in (p^*, p^* + \eta)$ would cause the equilibrium to unravel.

The same argument also kills all the hybrid equilibria. To see this, note that if the same type trades at different prices in equilibrium, all such prices should ensure the same expected payoff to the same type. Then, we can pick the payoff at a given price to represent the type's equilibrium payoff. Also, in any hybrid equilibrium, there

is at least one price, p^*, at which both qualities are trading. Hence, we just need to compare the payoffs at that price. Let $\sigma^*(h|p^*)$ be the probability to sample a high type at p^*. Condition (22) becomes:

$$u(h) - p > K^*(p^*)\{\sigma^*(h|p^*)[u(h) - p^*] + (1 - \sigma^*(h|p^*))[u(l) - p^*]\}, \qquad (69)$$

where $\sigma^*(h|p^*)$ is strictly lower than 1 by assumption. But then, any $p \in (p^*, p^* + \tilde{\eta})$, $\tilde{\eta} \equiv (1 - \sigma^*(h|p^*))K^*(p^*)[u(h) - u(l)] + (1 - K^*(p^*))u(h) > 0$ would cause the equilibrium to unravel. $\square$

Proof of lemma 8

Consider the incentive compatibility condition for buyers and low quality sellers (noting that the ICC for high quality sellers is never binding):

$$[p_l - v(l)] \geq [p_h - v(l)]J^*(p_h) \qquad (70)$$
$$[u(l) - p_l]K^*(p_l) = [u(h) - p_h]K^*(p_h), \qquad (71)$$

where we set $K^*(p_h) = 1$ whenever $p_h < u(h)$. We first show that if buyers make positive surplus, then $J^*(p_h) > 0$. Then, we prove that if $p_h > v(h)$ the ICC of low quality sellers holds with equality. Finally, using that result, we prove $J^*(p_h) > 0$ for the case in which buyers do not make any surplus at neither prices, i.e. $p_l = u(l)$ and $p_h = u(h)$ (buyers' ICC implies that they should experience the same expected surplus at both prices). When prices are below the buyers' reservation prices, no buyer ever chooses not to buy. But then, $K^*(p_l)$ and $J^*(p_h)$ must satisfy:

$$J^*(p_h) = \frac{\theta K^*(p_l) - (1 - \lambda)}{\lambda K^*(p_l)}. \qquad (72)$$

On the other hand, if $p_l < u(l)$, $K^*(p_l)$ must be 1. If $K^*(p_l)$ were less than 1, low quality sellers would increase their profits by announcing a price $p = p_l + \epsilon$, $\epsilon > 0$ slightly higher than p_l. This deviation would induce a buyer to buy, unless $p_l = u(l)$ since he would obtain the good at p with probability $1 > K^*(p_l)$. But then, $J^*(p_h) = \frac{\theta - (1 - \lambda)}{\lambda} > 0$.

We now show that, whenever $p_h > v(h)$, condition (32) implies that ICC_l holds with equality. Rewrite ICC_l as:

$$[p_l - v(l)] = [p_h - v(l)]J^*(p_h) + \eta \qquad (73)$$

for $\eta \geq 0$. Now substituting $p_l - v(l)$ in (32) yields

$$p_h \leq \frac{p_h[v(h) - v(l)] + v(h)\eta}{v(h) - v(l) + \eta}, \qquad (74)$$

which can be rewritten as

$$1 \leq \frac{v(h) - v(l) + \frac{v(h)}{p_h}\eta}{v(h) - v(l) + \eta}. \tag{75}$$

So, for $p_h > v(h)$, η must be zero.

We now turn to the case in which $p_l = u(l)$ and $p_h = u(h)$. Then, given that high quality sellers announce $p_h = u(h) > v(h)$ and given that low quality sellers' profits are $u(l) - v(l) > 0$, $J^*(p_h) > 0$ must be satisfied for the ICC$_l$ to hold with equality.

To conclude, we note that probabilities to buy and sell at equilibrium prices take only one possible ex-post realisation. Sellers always play pure strategies in equilibrium. Hence, the stochastic nature of the probabilities to buy and sell only depends on the number of buyers who buy at each of the equilibrium prices. Let X be the number of buyers who buy at p_l ex-post, and $B - X$ the number of buyers who buy at p_h. Accordingly, the ex-post probability to sell at p_l is $\min\left[\frac{X}{(1-\lambda)S}, 1\right]$. We know that the ex-ante expected value, $J^*(p_l)$, must equal 1. Hence, only realizations $X \geq (1-\lambda)S$ can take positive probability in equilibrium. Similarly, the ex-post probability to buy at p_l is $\min\left[\frac{(1-\lambda)S}{X}, 1\right]$. Again, the ex-ante expected value $K^*(p_l)$ must be one. But then, only realizations $X \leq (1-\lambda)S$ can take positive probability in equilibrium. Thus, the only possible realization of X is $X = (1-\lambda)S$. This implies that probabilities to buy and sell take a single value with certainty. Thus, either buyers play pure strategies, or they play mixed strategy but, in this case, the outcome of their joint randomization must converge to a certain value. In other words, there is no aggregate uncertainty.

□

Proof of lemma 9

Whenever $p_h > v(h)$, the ICC of type l holds with equality from lemma 8. But then, from buyers' ICC, p_l and p_h must satisfy

$$p_h = \frac{u(h) - K^*(p_l)[u(l) - v(l) + v(l)J^*(p_h)]}{1 - K^*(p_l)J^*(p_h)} \tag{76}$$

$$p_l = \frac{v(l) + J^*(p_h)[u(h) - v(l) - u(l)K^*(p_h)]}{1 - K^*(p_l)J^*(p_h)}. \tag{77}$$

We now verify when $K^*(p_l)$ and $J^*(p_h)$ are such that p_h and p_l satisfy the participation constraints: $v(l) \leq p_l \leq u(l)$, and $v(h) \leq p_h \leq u(h)$. $p_l \leq u(l)$ implies:

$$v(l) + J^*(p_h)[u(h) - v(l) - u(l)K^*(p_l)] \leq [1 - K^*(p_l)J^*(p_h)]u(l) \Rightarrow J^*(p_l) \leq \delta, \tag{78}$$

$p_l \geq v(l)$ implies:

$$J^*(p_h)[u(h) - v(l) - u(l)K^*(p_l)] \geq -K^*(p_l)J^*(p_h)v(l) \Rightarrow K^*(p_l) \leq \frac{1}{\delta}. \tag{79}$$

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Again $p_h \leq u(h)$ gives:

$$K^*(p_l)[u(l) - v(l) + v(l)J^*(p_h)] \leq -K^*(p_l)J^*(p_h)u(h) \Rightarrow J^*(p_l) \leq \delta. \qquad (80)$$

Finally, $p_h \geq v(h)$ yields:

$$u(h) - K^*(p_l)[u(l) - v(l) + v(l)J^*(p_h)] \leq [1 - K^*(p_l)J^*(p_h)]v(h), \qquad (81)$$

or:

$$u(h) - v(h) + K^*(p_l)[J^*(p_h)(v(h) - v(l)) - (u(l) - v(l))] \geq 0. \qquad (82)$$

Since $J^*(p_h) \leq \delta$ implies $J^*(p_h)(v(h) - v(l)) - (u(l) - v(l)) < 0$, the above expression becomes

$$K^*(p_l) \leq \frac{u(h) - v(h)}{u(l) - v(l) - J^*(p_h)(v(h) - v(l))}. \qquad (83)$$

Since $\delta < 1$, the condition $K^*(p_l) \leq \frac{1}{\delta}$ is always trivially satisfied for any $K^*(p_l)$. The condition $J^*(p_h) \leq \delta$ gives an upper bound for the probability to sell at the high price. Consider condition (83). As in the proof of lemma 8, we note that either $K^*(p_l) = 1$ or $p_l = u(l)$ (which, from the buyers' ICC, implies $p_h = u(h)$). If $K^*(p_l)$ were less than 1, low quality sellers would increase their profits by announcing a price $p = p_l + \epsilon$, $\epsilon > 0$ slightly higher than p_l. Unless $p_l = u(l)$, this deviation would induce a buyer to buy since he would obtain the good with probability 1 at p.

When $p_l = u(l)$ and $p_h = u(h)$, the participation constraint $p_h \geq v(h)$ is trivially satisfied. If $p_h < u(h)$, substituting $K^*(p_l) = 1$ in (83) gives $J^*(p_h) \geq \gamma$. If this condition did not hold, the price announced by the type h would not satisfy his participation constraint. In this case, the only possible equilibrium is that he announces $p_h = v(h)$, so that the ICC of type l does not need to hold with equality. We thus distinguish between the case in which $\gamma \leq 0$ and the case $\gamma > 0$.

Case 1: $\gamma \leq 0$. In this case, $J^*(p_h) \geq \gamma$ always holds. Then, equilibrium prices are either $p_l = u(l)$ and $p_h = u(h)$, which imply $J^*(p_h) = \delta$, or

$$p_h = v(l) + \frac{u(h) - u(l)}{1 - J^*(p_h)} \qquad (84)$$

$$p_l = v(l) + \frac{J^*(p_h)[u(h) - u(l)]}{1 - J^*(p_h)}. \qquad (85)$$

Since these prices leave positive surplus to the buyers, the equilibrium requires that no buyer chooses to stay out of the market. Therefore, $J^*(p_h)$ and $K^*(p_l)$ must satisfy the following:

$$J^*(p_h) = \frac{\theta K^*(p_l) - (1 - \lambda)}{\lambda K^*(p_l)}, \qquad (86)$$

which, for $K^*(p_l) = 1$, gives $J^*(p_h) = \frac{\theta - (1-\lambda)}{\lambda} \geq 0$. Substituting the value of $J^*(p_h)$ into (84) and (85) gives the equilibrium prices.

Thus, the high quality is always traded unless $\theta = 1 - \lambda$. Note now that $J^*(p_h) < \delta$ implies $\theta < 1 - \lambda + \lambda\delta$. Hence, we can have prices that leave positive surplus to the buyers only if the ratio of buyers to sellers, θ, is below the threshold $1 - \lambda + \lambda\delta$. What remains to show is that if $\theta < 1 - \lambda + \lambda\delta$, then $p_q = u(q)$, $q = h, l$, never holds. In other words, $K^*(p_l)$ must be 1. $p_h = u(h)$ always requires $J^*(p_h) = \delta$. Note that the maximum value that $J^*(p_h)$ can take in any case is $\frac{\theta - (1-\lambda)}{\lambda}$ which is the value that it takes if $K^*(p_l) = 1$ *and* all buyers are on the market (which is not necessarily true if $p_q = u(q)$, $q = h, l$). But then, given $\theta < 1 - \lambda + \lambda\delta$, $J^*(p_l) < \delta$ follows.

Case 2: $\gamma > 0$. If $\frac{\theta - (1-\lambda)}{\lambda} > \gamma$, then the condition $J^*(p_h) \geq \gamma$ holds and the above discussion applies. Prices are given by equations (84) and (85) for $\theta < 1 - \lambda + \lambda\delta$ and by $p_q = u(q)$ for $\theta \geq 1 - \lambda + \lambda\delta$. If, by converse, $\frac{\theta - (1-\lambda)}{\lambda} \leq \gamma$, type h would announce $v(h)$. From buyers' ICC, the price announced by type l is:

$$p_l = u(l) - K^*(p_l)[u(h) - v(h)] < u(l). \tag{87}$$

Again, since buyers make positive surplus, $K^*(p_l) = 1$ which implies $J^*(p_h) = \frac{\theta - (1-\lambda)}{\lambda}$. Note that $\gamma > 0$ is necessary for the type h to announce $v(h)$. On the other hand, $\gamma > 0$ always ensures that equation (87) does not violate $p_l \geq v(l)$.

Finally, note that since prices p_h and p_l and probabilities $J^*(p_h)$ and $K^*(p_l)$ only depend on parameters, the equilibrium outcome is unique. □

Proof of lemma 10

The result can be easily proved by analysing the behaviour of the equilibrium outcome as θ changes. From lemma 9, if $\theta \geq 1 - \lambda$, then the equilibrium outcome is unique. When $\theta < 1 - \lambda$, the high quality is not traded from lemma 8. In this case we can have either equilibria of type I where high quality sellers are out of the market or equilibria of type II in which high quality sellers announce a price at which no trade takes place. In both cases, price competition requires that low quality sellers make zero profits, so that trade only occurs at $p^* = v(l)$ and the quantity traded is always θS. □

Proof of lemma 11

Consider a deviation $p \in (v(q), p_q)$. Note that buyers are willing to buy at p if they think that the deviation comes from type q, since $p < p_q$. We argue that whenever it is possible to delete type $q - 1$ from the deviation it is also possible to delete all types $q - s$, $s \geq 2$. To show this point, assume that type $q - 1$ can be eliminated:

$$\frac{J(p_{q-1})[p_{q-1} - v(q-1)]}{p - v(q-1)} > \frac{J(p_q)[p_q - v(q)]}{p - v(q)}. \tag{88}$$

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Consider now type $q - s$. From the incentive compatibility condition:

$$\frac{J(p_{q-s})[p_{q-s} - v(q - s)]}{p - v(q - s)} \geq \frac{J(p_{q-1})[p_{q-1} - v(q - s)]}{p - v(q - s)}. \tag{89}$$

But then, for any $p > v(q) > p_{q-1}$, the following relationship

$$\frac{J(p_{q-s})[p_{q-s} - v(q - s)]}{p - v(q - s)} > \frac{J(p_{q-1})[p_{q-1} - v(q - 1)]}{p - v(q - 1)} \tag{90}$$

holds, which implies that type $q - s$ can be deleted, whenever type $q - 1$ can be deleted. Since $p < p_q < u(q) < v(q+1)$, sellers of type higher than q are never willing to deviate to p. This implies that if type $q - 1$ can be deleted, beliefs should be that the deviation comes from q. As in the two-quality case, we show that whenever $p_q > v(q)$, a viable deviation $p \in (v(q), p_q)$, for which type $q - 1$ can be eliminated, exists as long as the incentive compatibility condition of type $q - 1$ holds with inequality. Suppose then the ICC holds with strict inequality. Assume that the deviation consists in a price $p = p_q - \epsilon$, $\epsilon > 0$ which is a small undercutting of price p_q. We want to show that there exists $\epsilon > 0$ such that:

$$\frac{J(p_{q-1})[p_{q-1} - v(q - 1)]}{p_q - \epsilon - v(q - 1)} > \frac{J(p_q)[p_q - v(q)]}{p_q - \epsilon - v(q)}. \tag{91}$$

Condition (91) can be rewritten as:

$$\frac{J(p_{q-1})[p_{q-1} - v(q - 1)]}{J(p_q)[p_q - v(q - 1)]} > \frac{p_q - v(q)}{p_q - v(q - 1)} \frac{p_q - \epsilon - v(q - 1)}{p_q - \epsilon - v(q)}. \tag{92}$$

For $p_q > v(q)$, the LHS (which does not depend on ϵ) is strictly greater than 1 whenever the ICC of type $q - 1$ holds with strict inequality. On the other hand, the RHS goes to 1 as ϵ becomes small. Thus, there always exists ϵ such that type $q - 1$ can be deleted unless either the ICC holds with equality or $p_q = v(q)$. In fact, in the case $p_q = v(q)$, undercutting is never profitable for type q. Hence, either the ICC holds with equality or $p_q = v(q)$.

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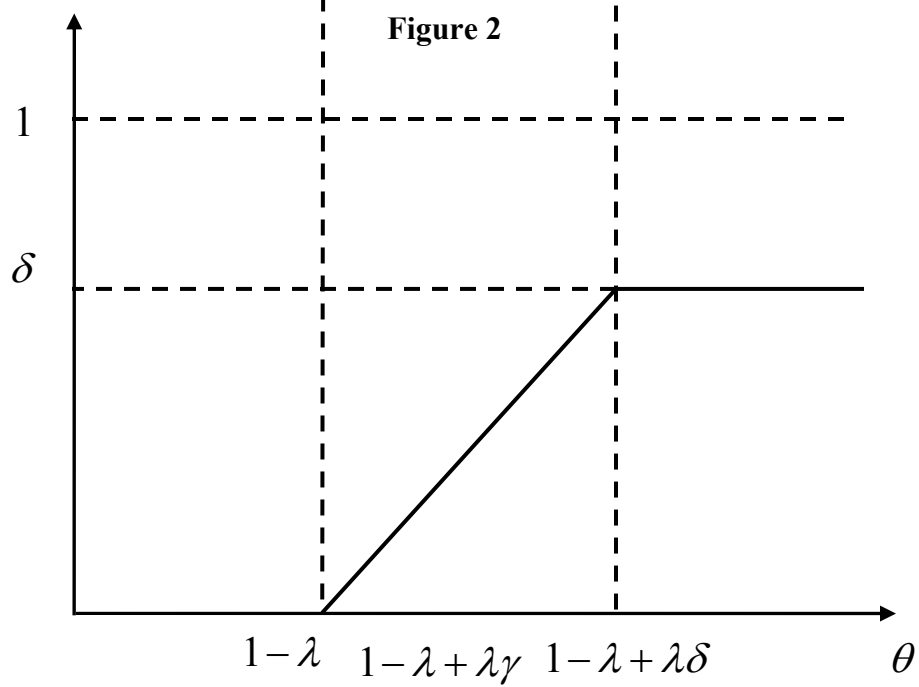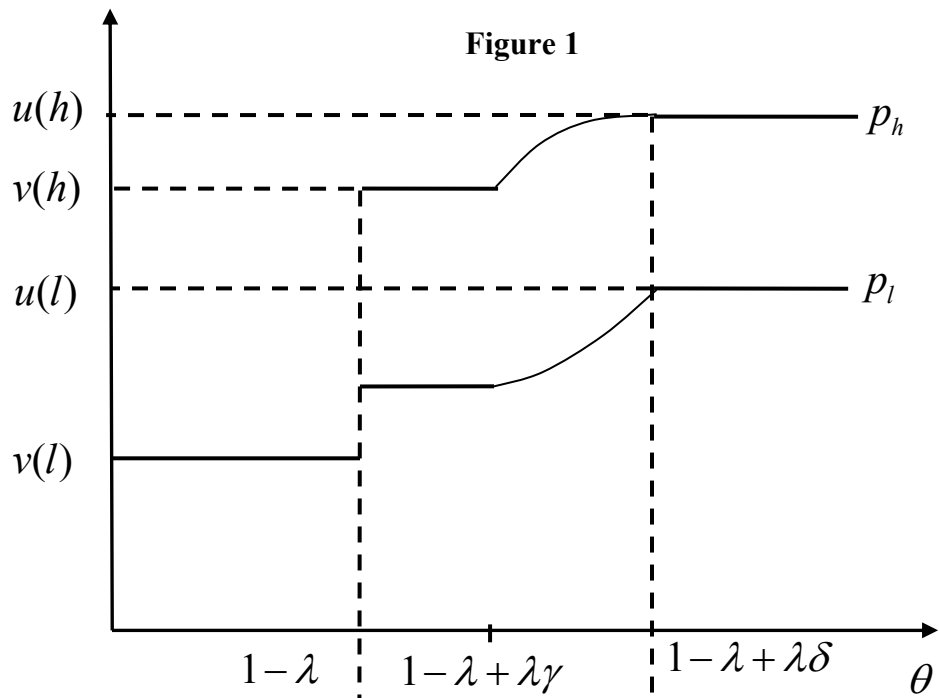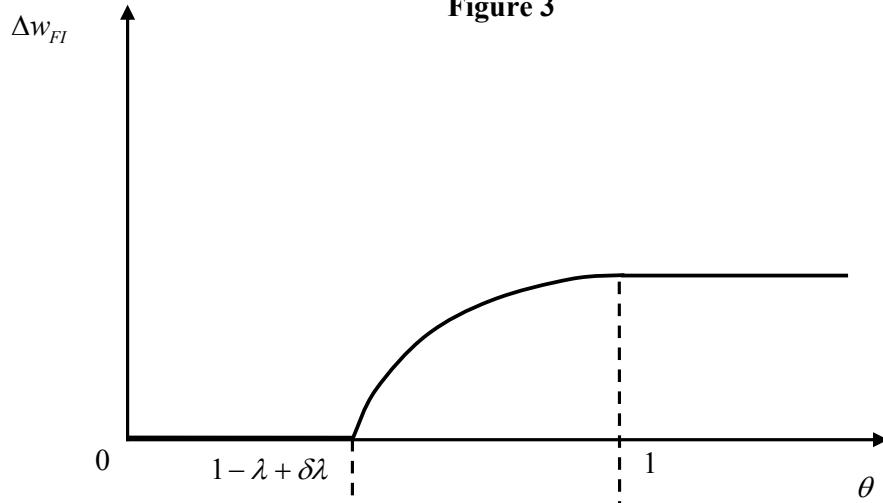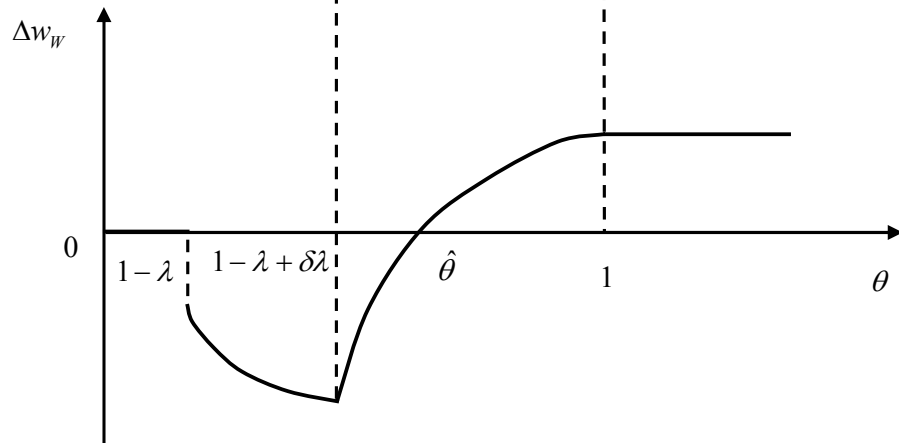
Figure 1

$u(h)$ ⋯⋯⋯⋯⋯⋯ p_h
$v(h)$
$u(l)$ ⋯⋯⋯⋯⋯⋯ p_l
$v(l)$

$1-\lambda$ $1-\lambda+\lambda\gamma$ $1-\lambda+\lambda\delta$ θ

Figure 2

1
δ

$1-\lambda$ $1-\lambda+\lambda\gamma$ $1-\lambda+\lambda\delta$ θ

Figure 3

Δw_{FI}

0 $1-\lambda+\delta\lambda$ 1 θ

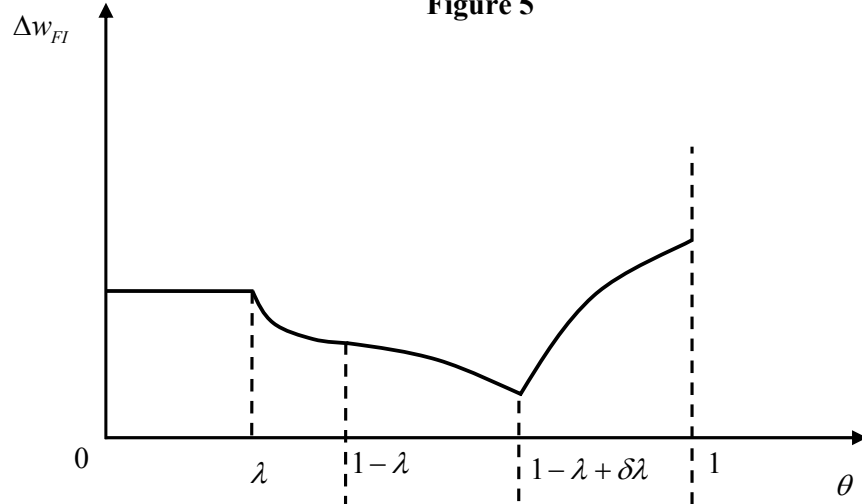**Figure 4**

Δw_W

0 $1-\lambda$ $1-\lambda+\delta\lambda$ $\hat{\theta}$ 1 θ

Figure 5

Figure 6