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THE RISK NEUTRAL VALUATION PARADOX

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WORKING PAPERS

2011/12

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Title: THE RISK NEUTRAL VALUATION PARADOX

ISBN: 978 88 8467 674 0

First Edition: July 2011

Via Is Mirrionis, 1
09123 Cagliari
Tel./Fax 070 291201
www.cuec.it

The risk neutral valuation paradox *

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Abstract

This paper highlights the role of risk neutral investors in generating endogenous bubbles in derivatives markets. We propose the following theorem. A market for derivatives, which has all the features of a perfect market except completeness and has some risk neutral investors, may exhibit almost surely extreme price movements which represent a violation to the Gaussian random walk hypothesis. This can be viewed as a paradox because it contradicts wide-held conjectures about prices in informationally efficient markets with rational investors. The theorem implies that prices are not always good approximations of the fundamental values of derivatives, and that extreme price movements like price peaks or crashes may have endogenous origin and happen with a higher-than-normal frequency. In the paper, we demonstrate the theorem and we propose an application that solves the Grossman-Stiglitz paradox on the value of information..

Keywords: risk neutral, martingale, derivatives, efficient market, fundamental theorem, bubble.

JEL Classification: C73, G12, G13 (primary); C78, G01 (secondary).

MSC 2000 Classification: 60G, 60H, 91B, 91G.

* I wish to thank Prof. Marco Vannini for being my mentor and Prof. Angelo Antoci for considerate guidance during my studies. I also wish to thank Dr Andrea Pinna and Dr Claudio Detotto for precious comments on seminal stages of the research. This work is part of a joint research program sponsored by the Department of Economics of the University of Sassari and the Regione Autonoma della Sardegna under the regional law n. 7/2007. I also gratefully acknowledge the support to research provided by the Doctoral School in Law and Economics of the University of Sassari.

1 Introduction

The present study addresses the role of risk neutral investors in generating turmoil in financial markets. Our hypothesis is that, in derivatives markets, random trades made by risk neutral investors and based on arbitrage free valuation can move prices away from their fundamental values. This effect can improve market instability by leading to abrupt price adjustments towards their fundamental values. These adjustments can take the form of either price peaks or price crashes, have an endogenous origin and can happen systematically. Our main findings are summarizad in the following theorem.

Theorem (risk neutral valuation paradox). *Let us consider an incomplete market for derivatives, which is frictionless, informationally efficient, and has competitive rational investors with different risk attitudes. Then, if some investors are risk neutral, the market may exhibit almost surely cycles in which prices diverge from their fundamental values, ended by abrupt price adjustments towards their fundamental values.*

The theorem suggests that derivatives markets with risk neutral investors can exhibit extreme price adjustments, which take the form of price peaks or crashes and represent a violation to the Gaussian random walk hypothesis. These adjustments may happen with a higher-than-normal frequency and may lead the distribution of price movements to become abnormal. A consequence of the theorem is that endogenous asset bubbles can happen almost surely in derivatives markets which are incomplete and have some risk neutral investors. It contradicts the conjecture that, in an informationally efficient market with competitive rational investors, prices should always be good approximations of the fundamental value of derivatives.

The theorem is a consequence of the fundamental theorem of asset pricing and the related martingale theory of bubbles. Following these theories, prices in a derivatives market can diverge from their fundamental values if and only if the market is incomplete. Thus, we intend to demonstrate our assumption in a market that has all the properties of a perfect market except completeness.

We define a competitive economy of pure exchange with uncertainty. We consider a single consumption good, which acts as numéraire. Agents are interested in certain consumption at present time and in state contingent consumption on future dates. There is an underlying asset, composed by the sequence of realizations $x_t(\omega)$ of the stochastic process X_t. A realization $x_t(\omega)$ represents the contingent consumption gained or lost at date t by the owner of the underlying asset if the state is ω. Tradable assets consist in the underlying asset and the set of its derivatives. We consider continuous trading dates. The market is incomplete so that price bubbles might come into existence.

Agents aim only at maximizing their expected utility. They differ for risk attitudes and can be divided into the two groups of risk averse and risk neutral. Risk averse agents will trade according to their subjective reservation prices for each tradable consumption bundle; we name this strategy as 'fundamental trading'. On the other hand, risk neutral agents can also trade according to the martingale equivalent probability measure that is inferable from market prices; we name this strategy as 'technical trading'.

Roughly speaking, the difference between the two strategies is the following. Fundamental traders aim at maximizing the income that accrues from their own portfolios in the form of contingent consumption claims, while technical traders aim also at maximizing trading capital gains. The two strategies can have the same expected return but the first is less risky than the second. Thus, risk averse agents will always be fundamental traders, while risk neutral agents will be indifferent between fundamental and technical trading and will switch randomly from one to the other.

We show that, in a market where all agents act as fundamental traders, the price process is *bounded*. We define such market as 'non speculative market'.

Once that an equilibrium price is reached, a martingale equivalent probability measure is inferable from the market and risk neutral agents can start to act as technical traders. We demonstrate that the entrance of technical traders in the market leads the price process to follow a *locally unbounded* Brownian motion; thus, there exists a positive probability that the market price may exit from the boundaries that previously existed when only fundamental traders were operating.

When the market price stays inside the boundaries of a non speculative market, we define the market conditions as 'normal'. When it goes outside these boundaries, we define the market conditions as either 'depression' or 'bubble', depending if the price stays, respectively, below or above the boundaries. In these cases, the market bears the risk of an abrupt price adjustment towards its fundamental values. Indeed, if risk neutral agents stop to act as technical traders, even for an instant, the market becomes again non speculative and the price goes back into the boundaries via an abrupt price adjustment. This will happen almost surely in a sufficiently long period of time, for the law of large numbers. These adjustments can take the form of either price peaks or crashes and their frequency will sum up to the normal price movements, thus affecting the tails of the distribution of price movements.

The paper starts with a brief exposition of the theoretical foundation of the theorem (sect. 2) and the mathematical formalization of the economic environment (sect. 3). The paper follows with the definition of the model and the demonstration of the theorem (sect. 4). Then, we propose a solution to the Grossman-Stiglitz paradox (sect. 5). Finally, conclusions are drawn (sect. 6).

2 Theoretical Foundation

The theorem that we propose is a consequence of what is known as the 'fundamental theorem of asset pricing', which is at the core of the arbitrage pricing theory. The fundamental theorem states that the absence of free lunch in a frictionless market is equivalent to the existence of an equivalent martingale probability measure, under which all assets have the same expected return, and that this measure is unique if and only if the market is complete.

The core of the modern theory of arbitrage pricing is based upon significant developments in stochastic calculus proposed by Itô and others between the 1930s and the early 1970s.[1] Its origins can also be

[1] We recall the fundamental contributions proposed by Kolmogorov on Markov processes [43] and on probability theory [44]; by Doeblin [18] and by Itô [33] [34] [35] on independent and identically distributed stochastic processes and their integration; by Doob [21] and Meyer [53] [54] on the decomposition theorem of mar-

traced to the modern theory of general equilibrium developed in the 1950s by Arrow, Debreu and McKenzie, and its successive evolutions.[2]

The foundations of the arbitrage pricing theory in modern[3] finance are provided by Black and Scholes [5] and by Merton [52] in their celebrated contributions on option valuation. Cox and Ross [9] introduce the concept of risk neutral valuation and argue that in a market with no arbitrage opportunities, it is possible to reassign probabilities to give all assets the same expected returns. Harrison and Kreps [29], Kreps [45] and Harrison and Pliska [30] make a breakthrough in arbitrage pricing theory and give a rigorous foundation to the fundamental theorem of asset pricing. They use Itô calculus to define the concept of no-arbitrage, refer to risk neutral probabilities as the 'equivalent martingale measure', and demonstrate the fundamental theorem for trading strategies that are simple integrands. These results were extended in various directions, among the others[4], by Dalang, Morton and Willinger [11], while Delbaen [15] and Schachermayer [65] prove the theorem for special cases.[5]

A general version of the fundamental theorem of asset pricing is provided by Delbaen and Schachermayer [16] [17]. They extend the theorem for trading strategies that are general integrands. Furthermore, they decline the no-arbitrage condition under the fundamental concept of 'no free lunch with vanishing risk' and they show that such condition is fulfilled if and only if there exists an equivalent probabil-

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tingales; by Girsanov [25] on the transformation of stochastic processes through a change of probability measure and numéraire; by Doléans-Dade [19] on the exponential processes of semimartingales and with Meyer [20] on the final version of the decomposition theorem; by Novikov [56] on the necessary and sufficient condition for a stochastic process to be a martingale.

[2] For the theoretical foundation of the modern theory of general equilibrium, see Arrow and Debreu [2], Debreu [12] and McKenzie [51]. Important advances in the theory come from Sonnenschein in [69] and [70], Mantel in [50] and Debreu in [13].

[3] Seminal works on derivatives pricing were already proposed, at the turn of the twentieth century, by Regnault [60], Bachelier [3], Nelson [55] and Bronzin [6].

[4] See also: Dybvig and Huang [23], Duffie and Huang [22], Stricker [71], Ansel and Stricker [1] and Lakner [47].

[5] Simple proofs are also proposed by Schachermayer [64], Kusuoka [46], Kabanov and Kramkov [41] and Rogers [61].

ity measure under which the price process is a sigma martingale or, in the continuous case, a local martingale.

The literature and theories discussed above are at the base of what is sometimes called the 'martingale theory of bubbles'.[6] Harrison and Kreps [28] provide the classical definition of the fundamental value of an asset as the value of its discounted cash flows under the equivalent martingale measure. Therefore, a bubble would originate from the deviation between the fundamental value of an asset and its market price. This theory focuses on the characteristics of asset bubbles and the evaluation of derivative securities in economies which satisfy the no arbitrage condition.

The martingale approach requires considerably looser conditions than the theory of bubbles in the classical economic framework. Indeed, in economic equilibrium models, the price function requires a precise definition of additional characteristics of the economy, such as the functions of supply and demand.[7]

In the martingale theory of bubbles, the no arbitrage condition is often imposed in the form of 'no free lunch with vanishing risk' (NFLVR) proposed by Delbaen and Schachermayer [16]. In the models proposed by Loewenstein and Willard [48] [49], by Cox and Hobson [8] and by Heston, Loewenstein and Willard [32], bubbles violate numerous classical option pricing theorems including the put-call parity. This result is only partially supported by empirical evidence.[8] To overcome this limitation, Jarrow, Protter and Shimbo [39] [40] impose the further condition of no dominance[9] of portfolios. This condition is stronger that the NFLVR but still substantially weaker than imposing a market equilibrium. They show that the addition of the no dominance condition excludes all asset price bubbles in complete markets with infinite trading horizons, because the fundamental values of as-

[6] See for example Jarrow and Protter [38].

[7] We recall the contributions on asset bubbles in classical economics proposed by Tirole [72] and Santos and Woodford [62] on markets with finite trading horizon and rational expectations, by Tirole [73], O'Connell and Zeldes [57] and Weil [75] on markets with rational traders and infinite trading horizon, and by De Long et al. [14] on markets where there are irrational traders. For good reviews, see Camerer [7], Scheinkman and Xiong [66], and Jarrow, Protter and Shimbo [40].

[8] See for example Kamara and Miller [42] and Ofek and Richardson [58].

[9] See Merton [52].

sets and their market prices should always be identical. Consequently, if bubbles are to exist, the market should be incomplete.

According to the fundamental theorem of asset pricing, an incomplete market presents a wide range of different martingale measures that may be used for estimating the fundamental values of assets. Eberlein and Jacod [24] suggest that these martingale measures and, consequently, the prices of derivatives should have a closed range of variation. Schweizer and Wissel [67] and Jacod and Protter [36] argue that market prices of derivatives can reveal which one of these martingale measures is the one currently adopted by the market. Then, a bubble can start (from a situation of non-bubble) when traders decide to adopt a different martingale measure. In economic terms, this variation will correspond to a regime change in the core values of the economy (endowments, beliefs, risk aversion, technologies, institutional structures). In financial terms, the change in the martingale measure can leave unchanged the price of the underlying security but, as the market is incomplete, it will change the price of some of its derivative securities. To investigate the birth of a bubble or its presence in the market is therefore essential to analyze temporal trends in the prices of derivative securities[10] as we do in this paper.

The contribute of this paper to the theory of financial markets is to highlight the role of risk neutral investors in generating endogenous bubbles in derivatives markets. In the following sections, we propose and demonstrate a theorem of mathematical finance that can help to explain how extreme price movements like price peaks or crashes, which represent a violation to the Gaussian random walk hypothesis, can originate endogenously and may happen with higher-than-normal frequency. Moreover, we extend the theorem to the case of asymmetric information and we propose a solution to the Grossman-Stiglitz paradox.

[10] See also Jarrow, Kchia and Protter [37].

3 The Economic Environment

3.1 Basic Definitions

Let us consider a competitive economy of pure exchange with uncertainty. We are given a continuous set of trading dates $t \in [0,\ T]$ where t are the instants of time at which market participants can trade. There is a single consumption good, which acts as numéraire.

Agents are interested in certain consumption at present time and in state contingent consumption on future dates. Future consumption is represented by functions of the underlying asset $\tilde{\mathbf{x}}_t$: this is the vector composed by the sequence of realizations $x_t(\omega)$ of the stochastic process X_t. The realization $x_t(\omega)$ represents the contingent consumption gained or lost at date t by the owner of the underlying asset $\tilde{\mathbf{x}}_t$ if the state is ω. The underlying asset and its possible derivative functions form the convex set $\mathbf{x}_t^{set}$, which represents the family of tradable vectors $\mathbf{x}_t$ that can be created and exchanged at date t. In other words, $\mathbf{x}_t^{set}$ represents the securities traded in the market at any given date t. To improve readability, we will write $\mathbf{x}_t$ for the generic element of $\mathbf{x}_t^{set}$.

At each trading date t agents can exchange any vector of future contingent consumption $\mathbf{x}_t \in \mathbf{x}_t^{set}$ with units of certain present consumption r_t. Thus, we consider consumption bundles of the form $(r_t, \mathbf{x}_t)$ where the real number r_t represents the units of certain consumption at the present date t and the vector $\mathbf{x}_t$ represents the units of state contingent consumptions over the interval $(\,t,\ T\,]$.

3.2 Probability Assumptions

Let $(X_t)_{t\in[0,T]}$ be an adapted càdlàg stochastic process on the probability space $(\,\Omega,\ \mathbb{F},\ \mathbb{P}\,)$, where:

- the universal sample space Ω is the set of all possible elementary outcomes $\omega \in \Omega$;
- the filtration $\mathbb{F}$ is the set of the σ-algebras $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, where $\mathcal{F}_t$ represents the information available at any time t;
- the probability measure $\mathbb{P}$ is the set of measures $\mathbb{P} = \{\mathcal{P}_t\}_{t\in[0,T]}$, where $\mathcal{P}_t$ is the probability measure of X_t at a given date t according to the available information $\mathcal{F}_t$.

The underlying probability space that we define fulfills the usual hypotheses of the general theory of financial asset pricing. We can interpret $\mathcal{F}_t$ as the questions that agents can answer at time t regarding past and present states of the world. The information becomes more and more precise (i.e. the set of measurable events increase) as new events from the present becomes known.

For the purpose of the model, we assume that the information is continuously and completely available to each individual (and, accordingly, that the filtration is right continuous). Furthermore, we assume that information is free. Thus, no additional costly information is available and the model avoids any potential application of the Grossman and Stiglitz paradox [26] on the impossibility of informationally efficient markets.[11] We also assume, with no loss of relevance, that investors have common prior beliefs on the probability of future events.

3.3 Market Participants

We define rationality in strictly axiomatic form, according to von Neumann and Morgenstern [74] and Savage [63]. Furthermore, we assume that all the individuals operating in the market are rational.

We denote the set of conceivable agents with $\mathbf{A}$. The i-th agent is characterized at each date t by an initial endowment $(\hat{r}_t^i, \hat{\mathbf{x}}_t^i)$ and a preference relation $\succsim_i$ over the space of consumption bundles $(r_t, \mathbf{x}_t)$. Preferences depend on the subjective degree of risk aversion and are assumed to be complete, transitive, convex, continuous, strictly increasing and independent.Following these properties, the agent is able to express the preferences towards consumption bundles via an additive von Neumann-Morgenstern utility function $u_i : (r_t, \mathbf{x}_t) \to \mathbb{R}$.

Agents aim only at maximizing their expected utility, and choose the strategies that they repute best performing. There are two possible alternative strategies: *fundamental* trading and *technical* trading. For improving the clarity of the demonstration, we assume with no loss of relevance that both these strategies are simple integrands.

[11] The case of asymmetric information, with a proposed solution to the Grossman-Stiglitz paradox, will be considered in sect. 5.

Roughly speaking,[12] the difference between the two strategies is the following. Fundamental traders aim at maximizing the income that accrues from their own portfolios in the form of contingent consumption claims, while technical traders aim also at maximizing trading capital gains.

The two strategies can have the same expected return but the first is less risky than the second. Thus, risk averse agents will always prefer fundamental trading. On the other hand, risk neutral agents will be indifferent between fundamental and technical trading and will switch randomly from one strategy to the other.

Although risk neutral agents can operate as both fundamental and technical traders, to improve the clarity of presentation, we prefer to consider them as distinct individuals: a fundamental trader and a technical trader which operate alternatively and alternately at random times. In the same way, although each agent can both buy and sell, we prefer to consider each fundamental trader as two different individuals, one that exclusively buy and the other that exclusively sell. Then, from the intersection of the different categories, we can formally define the set of conceivable agents $\mathbf{A} : \{\mathbf{FB}, \mathbf{FS}, \mathbf{TB}, \mathbf{TS}\}$ as composed by the distinct subgroups of fundamental traders that are buyers, $\mathbf{FB}$, and sellers, $\mathbf{FS}$, and by the technical traders that are buyers, $\mathbf{TB}$, and sellers, $\mathbf{TS}$.

As said above, the number of technical traders varies randomly over time depending on the choices of risk neutral investors between indifferent alternatives. Thus, we will also use the notation $\mathbf{TB}_t$ and $\mathbf{TS}_t$ to denote the sets of risk neutral agents acting as technical traders (respectively, buyers and sellers) at a specific time t.

3.4 Trading System

Exchanges happen within an electronic trading system. Agents submit orders to the electronic system, which immediately searches for matching orders; in case they exist, the system executes the trade and remits the numéraire to the traders. To be more precise, when a consumption bundle $(r_t, \mathbf{x}_t)$ is traded, the system remits immediately the

[12] A more rigorous definition is given in section 4, propositions 5 and 6 and definition 4.

payment r_t to the seller and, from then inwards, it pays or subtracts to the buyer (as it accrues) the contingent consumption generated by $\mathbf{x}_t$. Agents go bankrupt and must exit from the market if their wealth represented by units of certain consumption is wiped out by losses.

Agents' trading orders can be of four types.

- The limited order of buying, $Bid_i(\mathbf{x}_t)$, which expresses the *maximum* amount of certain present consumption that an agent $i \in \mathbf{FB}$ is willing to offer for buying a vector of future contingent consumption $\mathbf{x}_t$.
- The limited order of selling, $Ask_i(\mathbf{x}_t)$, which expresses the *minimum* amount of certain present consumption that an agent $i \in \mathbf{FS}$ is willing to accept for selling a vector of future contingent consumption $\mathbf{x}_t$.
- The market order of buying, $Buy_i(\mathbf{x}_t)$, which expresses the will of an agent $i \in \mathbf{TB}$ to buy a given vector of future contingent consumption $\mathbf{x}_t$ at *any* price, as long as it is the market-clearing price $Equil\,(\mathbf{x}_t)$ plus a small sum ϵ.
- The market order of selling, $Sell_i(\mathbf{x}_t)$, which expresses the will of an agent $i \in \mathbf{TS}$ to sell a given vector of future contingent consumption $\mathbf{x}_t$ at *any* price, as long as it is the market-clearing price $Equil\,(\mathbf{x}_t)$ minus a small sum ϵ.

We assume that the market orders of buying and selling, $Buy(\mathbf{x}_t)$ and $Sell(\mathbf{x}_t)$, put, respectively, upward and downward pressure on the market-clearing price, $Equil\,(\mathbf{x}_t)$. The final effect on the price at time $t + dt$ depends on which of the two sets of technical traders, buyers $\mathbf{TB}$ or sellers $\mathbf{TS}$, is larger at time t.

The element ϵ can be interpreted as the Bayesian updating of price expectations of technical agents due to their choice of buying or selling. We can also interpret ϵ as a small sum that agents should add or subtract to the market-clearing price $Equil\,(\mathbf{x}_t)$ to make sure that their market orders of buying or selling, $Buy(\mathbf{x}_t)$ and $Sell(\mathbf{x}_t)$, would be executed.

3.5 Price process

The market-clearing price, *Equil* $(\mathbf{x}_t)$, maximizes the quantities exchanged in the market and is determined according to the price function:

$$Equil(\mathbf{x}_t) := f\Big\{Bid_{\mathrm{FB}}(\mathbf{x}_t),\ Ask_{\mathrm{FS}}(\mathbf{x}_t),\\ Buy_{\mathrm{TB}}(\mathbf{x}_t),\ Sell_{\mathrm{TS}}(\mathbf{x}_t),\ Equil(\mathbf{x}_{t-dt})\Big\} \mapsto \mathbb{R}, \quad (1)$$

where $Bid_{\mathrm{FB}}(\mathbf{x}_t)$ and $Ask_{\mathrm{FS}}(\mathbf{x}_t)$ are the sets of trading orders from fundamental traders that are, respectively, buyers and sellers, and where $Buy_{\mathrm{TB}}(\mathbf{x}_t)$ and $Sell_{\mathrm{TS}}(\mathbf{x}_t)$ are the sets of trading orders from technical traders that are, respectively, buyers and sellers.

4 The Model

In the present section, we intend to demonstrate that, when no risk neutral investor is operating as technical trader, the price function in formula (1) becomes bounded. On the other hand, when some risk neutral investors are operating in the market as technical traders, the price function follows a locally unbounded Brownian motion.

Then, the demonstration of the risk neutral valuation paradox comes from the alternation of boundedness and unboundedness of the price process, due to risk neutral investors who switch randomly over time from fundamental to technical trading and vice versa.

4.1 Basic definitions

Definition 1 (expected utility). *The expected utility of a given consumption bundle* $(r_t,\ \mathbf{x}_t)$ *for the i-th agent is the function:*

$$U_i(r_t, \mathbf{x}_t) = E\left[\, u_i(r_t, \mathbf{x}_t)\,\right] = \int_{\Omega} u_i\left(\, r_t,\ \mathbf{x}_t(\omega)\,\right) \mathcal{P}_t\, d\omega, \quad \forall\, i \in \mathbf{A}, \quad (2)$$

where $\mathcal{P}_t \in \mathbb{P}$ *is the probability measure given the information* $\mathcal{F}_t$*.*

Lemma 1 (preference between consumption bundles). *Let* $(\,\dot{r}_t,\ \dot{\mathbf{x}}_t\,)$ *and* $(\,\ddot{r}_t,\ \ddot{\mathbf{x}}_t\,)$ *be two different consumption bundles. The i-th agent strictly* *prefers the first bundle to the second if and only if:*

$$(\dot{r}_t, \dot{\mathbf{x}}_t) \succ_i (\ddot{r}_t, \ddot{\mathbf{x}}_t) \iff U_i(\dot{r}_t, \dot{\mathbf{x}}_t) > U_i(\ddot{r}_t, \ddot{\mathbf{x}}_t), \quad \forall i \in \mathbf{A}. \tag{3}$$

The i-th agent is indifferent between the two consumption bundles if and only if:

$$(\dot{r}_t, \dot{\mathbf{x}}_t) \sim_i (\ddot{r}_t, \ddot{\mathbf{x}}_t) \iff U_i(\dot{r}_t, \dot{\mathbf{x}}_t) = U_i(\ddot{r}_t, \ddot{\mathbf{x}}_t), \quad \forall i \in \mathbf{A}. \tag{4}$$

4.2 Market with only fundamental traders

Proposition 1 (strategy of fundamental traders). *A seller $i \in \mathbf{FS}$ with endowment $(\hat{r}^i_t, \hat{\mathbf{x}}^i_t)$ at date t will be willing to receive for a given vector of future contingent consumption $\mathbf{x}_t$ no less than the reservation price $Ask_i(\mathbf{x}_t)$, such that:*

$$\begin{aligned} U_i(\hat{r}^i_t + Ask_i(\mathbf{x}_t), \hat{\mathbf{x}}^i_t - \mathbf{x}_t) &= U_i(\hat{r}^i_t, \hat{\mathbf{x}}^i_t), \\ E[\mathbf{x}_t] > 0, \qquad Ask_i(\mathbf{x}_t) &> 0, \qquad \forall i \in \mathbf{FS}. \end{aligned} \tag{5}$$

A buyer $j \in \mathbf{FB}$ with endowment $(\hat{r}^j_t, \hat{\mathbf{x}}^j_t)$ at date t will be willing to pay for a given vector of future contingent consumption $\mathbf{x}_t$ no more than the reservation price $Bid_j(\mathbf{x}_t)$, such that:

$$\begin{aligned} U_j(\hat{r}^j_t - Bid_j(\mathbf{x}_t), \hat{\mathbf{x}}^j_t + \mathbf{x}_t) &= U_j(\hat{r}^j_t, \hat{\mathbf{x}}^j_t), \\ E[\mathbf{x}_t] > 0, \qquad Bid_j(\mathbf{x}_t) &> 0, \qquad \forall j \in \mathbf{FB}. \end{aligned} \tag{6}$$

Proof. According to rationality axioms, preferences are strictly increasing:

$$(\hat{r}^i_t, \hat{\mathbf{x}}^i_t) \succ_i (\hat{r}^i_t, \hat{\mathbf{x}}^i_t - \mathbf{x}_t), \qquad E[\mathbf{x}_t] > 0, \qquad \forall i \in \mathbf{A}. \tag{7}$$

For the property of continuity in preference relations, no consequence is infinitely better or infinitely worse than any other. Thus, if we add a positive amount of certain current consumption ϵ to the right-hand side of the relation, there exists one and only one value of ϵ such that the i-th agent is indifferent between the two consumption bundles:

$$\exists\, \epsilon > 0 \quad \Longrightarrow \quad (\, \hat{r}^i_t,\, \hat{\mathbf{x}}^i_t\,) \sim_i (\, \hat{r}^i_t + \epsilon,\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,). \tag{8}$$

Let us denote $\epsilon = Ask_i\,(\mathbf{x}_t)$. Then:

$$\exists\, \epsilon > 0 \quad \Longrightarrow \quad (\, \hat{r}^i_t,\, \hat{\mathbf{x}}^i_t\,) \sim_i (\, \hat{r}^i_t + Ask_i\,(\mathbf{x}_t),\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,), \tag{9}$$

which, according to lemma 1, is equivalent to formula (5) in proposition 1:

$$U_i\,(\, \hat{r}^i_t,\, \hat{\mathbf{x}}^i_t\,) \;=\; U_i\,(\, \hat{r}^i_t + Ask_i\,(\mathbf{x}_t),\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,). \tag{10}$$

The proof of formula (6) logically follows from that of formula (5). □

Corollary 1. *The seller $i \in \mathbf{FS}$ will strictly prefer any price higher than $Ask_i\,(\mathbf{x}_t)$:*

$$U_i\,(\, \hat{r}^i_t + Ask_i(\mathbf{x}_t) + \epsilon,\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,) \;>\; U_i\,(\, \hat{r}^i_t + Ask_i(\mathbf{x}_t),\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,),$$
$$E[\mathbf{x}_t] > 0, \qquad Ask_i(\mathbf{x}_t) > 0, \qquad \epsilon > 0, \qquad \forall\, i \in \mathbf{FS}. \tag{11}$$

The buyer $j \in \mathbf{FB}$ will strictly prefer any price lower than $Bid_j\,(\mathbf{x}_t)$:

$$U_j\,(\, \hat{r}^j_t - Bid_j\,(\mathbf{x}_t) + \epsilon,\ \hat{\mathbf{x}}^j_t + \mathbf{x}_t\,) \;>\; U_j\,(\, \hat{r}^j_t - Bid_j\,(\mathbf{x}_t),\ \hat{\mathbf{x}}^j_t + \mathbf{x}_t\,),$$
$$E[\mathbf{x}_t] > 0, \qquad Bid_j(\mathbf{x}_t) > 0, \qquad \epsilon > 0, \qquad \forall\, j \in \mathbf{FB}. \tag{12}$$

Proof. According to rationality axioms, preference relations are continuous and strictly increasing:

$$(\, \dot{r}_t + \epsilon,\, \dot{\mathbf{x}}_t\,) \succ_i (\, \dot{r}_t,\, \dot{\mathbf{x}}_t\,), \qquad \epsilon \;>\; 0, \qquad \forall\, i \in \mathbf{A}. \tag{13}$$

Let us denote $\dot{r}_t = \hat{r}^i_t + Ask_i(\mathbf{x}_t)$, and $\dot{\mathbf{x}}_t = \hat{\mathbf{x}}^i_t - \mathbf{x}_t$. Then:

$$(\, \hat{r}^i_t + Ask_i(\mathbf{x}_t) + \epsilon,\, \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,) \succ_i (\, \hat{r}^i_t + Ask_i(\mathbf{x}_t),\, \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,), \tag{14}$$

which, according to lemma 1, is equivalent to formula (11):

$$U_i\,(\, \hat{r}^i_t + Ask_i(\mathbf{x}_t) + \epsilon,\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,) \;>\; U_i\,(\, \hat{r}^i_t + Ask_i(\mathbf{x}_t),\ \hat{\mathbf{x}}^i_t - \mathbf{x}_t\,). \tag{15}$$

The proof of formula (12) logically follows from that of formula (11). □

Corollary 2. *The exchange of a vector* $\mathbf{x}_t$ *between a seller* $i \in \mathbf{FS}$ *and a buyer* $j \in \mathbf{FB}$ *can happen only at a price bounded between the reservation prices of the seller,* $Ask_i(\mathbf{x}_t)$*, and of the buyer,* $Bid_j(\mathbf{x}_t)$*:*

$$\exists\, Equil(\mathbf{x}_t) \;\Rightarrow\; Ask_i(\mathbf{x}_t) \;\leq\; Equil(\mathbf{x}_t) \;\leq\; Bid_j(\mathbf{x}_t), \\ \forall\, i \in \mathbf{FS} \;\wedge\; \forall\, j \in \mathbf{FB}. \tag{16}$$

Proof. A necessary condition for the exchange is that both seller and buyer prefer to their initial endowment the consumption bundle resulting from the trade:

$$\exists\, Equil(\mathbf{x}_t) \Rightarrow \begin{cases} (\,\hat{r}_t^i + Equil(\mathbf{x}_t),\, \hat{\mathbf{x}}_t^i - \mathbf{x}_t\,) \succsim_i (\,\hat{r}_t^i,\, \hat{\mathbf{x}}_t^i\,), \quad \forall\, i \in \mathbf{FS} \\ (\,\hat{r}_t^j - Equil(\mathbf{x}_t),\, \hat{\mathbf{x}}_t^j + \mathbf{x}_t\,) \succsim_j (\,\hat{r}_t^j,\, \hat{\mathbf{x}}_t^j\,),\ \forall j \in \mathbf{FB}. \end{cases} \tag{17}$$

From formulae (5) and (6), the condition is equivalent to corollary 2:

$$\exists\, Equil\,(\mathbf{x}_t) \Longrightarrow \begin{cases} Equil\,(\mathbf{x}_t) \;\geq\; Ask_i\,(\mathbf{x}_t),\ \forall\, i \in \mathbf{FS} \\ Equil\,(\mathbf{x}_t) \;\leq\; Bid_j\,(\mathbf{x}_t),\ \forall\, j \in \mathbf{FB}.\ \square \end{cases} \tag{18}$$

Proposition 2 (boundedness of price processes). *In a market with only fundamental traders, the price process of any tradable vector* $\mathbf{x}_t \in \mathbf{x}_t^{set}$ *is bounded. Its market-clearing price,* $Equil(\mathbf{x}_t)$*, assumes values in the closed interval between the minimum reservation price among the sellers and the maximum reservation price among the buyers, and exists if and only such interval exists and is positive:*

$$\exists\, Equil(\mathbf{x}_t) \iff \min[Ask_{\mathrm{FS}}(\mathbf{x}_t)] \;\leq\; Equil(\mathbf{x}_t) \;\leq\; \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)], \\ \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)] \;\geq\; \min[Ask_{\mathrm{FS}}(\mathbf{x}_t)]. \tag{19}$$

Proof. We proceed *per absurdum*. It results:

$$Equil(\mathbf{x}_t) \;<\; \min[Ask_{\mathrm{FS}}(\mathbf{x}_t)] \;\Rightarrow\; \nexists i \in \mathbf{FS}\colon\ Equil(\mathbf{x}_t) \;\geq\; Ask_i(\mathbf{x}_t), \tag{20}$$

and:

$$Equil(\mathbf{x}_t) \ > \ \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)] \ \Rightarrow \ \nexists j \in \mathbf{FB} \colon Equil(\mathbf{x}_t) \ \leq \ Bid_j(\mathbf{x}_t), \tag{21}$$

which violate, respectively, the first and the second conditions in formula (30). Thus, formula (19) is a necessary condition for the existence of the market-clearing price. The condition is also sufficient because, according to formula (1), it results:

$$\begin{aligned} \exists\, f \colon Ask_{\mathrm{FS}}(\mathbf{x}_t) \times Bid_{\mathrm{FB}}(\mathbf{x}_t) \ &\mapsto \ Equil(\mathbf{x}_t) \in \mathbb{R}, \\ \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)] \ &\geq \ \min[Ask_{\mathrm{FS}}(\mathbf{x}_t)]. \end{aligned} \tag{22}$$

Now, we demonstrate that the price process of each tradable vector is bounded. Let us suppose *per absurdum* that:

$$\exists\, Equil(\mathbf{x}_t) \ \notin \ \Big[\ \min\,[Ask_{\mathrm{FS}}(\mathbf{x}_t)], \ \max\,[Bid_{\mathrm{FB}}(\mathbf{x}_t)] \Big]. \tag{23}$$

Then, at least one of the following must be true:

$$Equil(\mathbf{x}_t) \geq Ask_i(\mathbf{x}_t), \qquad \nexists\, i \in \mathbf{FS}. \tag{24}$$

$$Equil(\mathbf{x}_t) \leq Bid_j(\mathbf{x}_t), \qquad \nexists\, j \in \mathbf{FB}. \tag{25}$$

$$\mathbf{x}_t \notin \mathbf{x}_t^{set}. \tag{26}$$

The first and the second propositions cannot be true because violate, respectively, the first and the second conditions in formula (30); the latter cannot be true by definition. □

Definition 2. *We define the space of Pareto-efficient bargains, $\mathbf{E}_t$, as the set:*

$$\mathbf{E}_t := \Big\{ \ (r_t,\, \mathbf{x}_t) \ \in \Big[\ \min\,[Ask_{\mathrm{FS}}(\mathbf{x}_t)], \ \max\,[Bid_{\mathrm{FB}}(\mathbf{x}_t)] \Big] \ \times \ \mathbf{x}_t^{set} \ \Big\}. \tag{27}$$

Any element within the space of Pareto-efficient bargains $\mathbf{E}_t$ denote a price r_t for the asset $\mathbf{x}_t$, at which at least one seller and one buyer (both fundamental traders) may find convenient to make the exchange. The set $\mathbf{E}_t$ can also be considered as the (closed) codomain of the function Equil $(\mathbf{x}_t^{set})$.

Let us define an arbitrage opportunity as a trading strategy that does not require the investment of current consumption, has a positive probability of gaining additional consumption and cannot lead to consumption losses. Then, the following proposition holds.

Proposition 3. *In a market with only fundamental traders, there are no arbitrage opportunities.*

Proof. By assumption, the risk free interest rate in the market is zero. Then, in order to have an arbitrage opportunity, there must exists: a security that has more than one price in the market, or some replicating portfolios ϕ_t and ϕ'_t with identical cash flows and different market-clearing prices. Let us demonstrate *per absurdum* that this is not possible. Let us assume that:

$$\exists\, \mathbf{x}_t,\, \mathbf{x}'_t \in \mathbf{x}_t^{set} \;:\; Equil(\mathbf{x}_t) \neq Equil(\mathbf{x}'_t) \;\wedge\; \mathbf{x}_t = \mathbf{x}'_t. \tag{28}$$

Then, according to formula (1), it should result:

$$f: Ask_{\mathrm{FS}}(\mathbf{x}_t) \times Bid_{\mathrm{FB}}(\mathbf{x}_t) \mapsto Equil(\mathbf{x}_t) \neq \\ \neq f: Ask_{\mathrm{FS}}(\mathbf{x}'_t) \times Bid_{\mathrm{FB}}(\mathbf{x}'_t) \mapsto Equil(\mathbf{x}'_t), \tag{29}$$

which for $Ask_{\mathrm{FS}}(\mathbf{x}_t) \times Bid_{\mathrm{FB}}(\mathbf{x}_t) = Ask_{\mathrm{FS}}(\mathbf{x}'_t) \times Bid_{\mathrm{FB}}(\mathbf{x}'_t)$ is clearly false.

The proof for replicating portfolios with identical cash flows, ϕ_t and ϕ'_t, logically follows from the preceding given that, for the transitivity in preferences, it results:

$$\phi_t \approx \phi'_t \implies \\ \Rightarrow \begin{cases} Ask_i(\phi_t) = Ask_i(\phi'_t),\ \forall i \in \mathbf{FS} \Leftrightarrow Ask_{\mathrm{FS}}(\phi_t) = Ask_{\mathrm{FS}}(\phi'_t) \\ Bid_j(\phi_t) = Bid_j(\phi'_t),\ \forall j \in \mathbf{FB} \Leftrightarrow Bid_{\mathrm{FB}}(\phi_t) = Bid_{\mathrm{FB}}(\phi'_t). \end{cases} \quad \square \tag{30}$$

Please note that the absence of arbitrage and free lunch opportunities (as in proposition 3) can be equivalent to the existence of an equivalent martingale measure $\mathcal{Q}_t$, under which the price processes of all securities become a martingale, according to the fundamental theorem of asset pricing and under some topological conditions.[13] In a market with only fundamental traders, these conditions, which we enumerate in the following, are all satisfied:

- existing market-clearing prices of securities are unique;
- existing market-clearing prices of replicating portfolios have equal value;
- agents' have a lower bound on wealth;
- admissible trading strategies are simple integrands;
- the price process of each asset $\mathbf{x}_t$ is bounded.

4.3 Market with fundamental and technical traders

In this section we demonstrate that, in a market with fundamental traders, there exists a martingale equivalent probability measure that permits the risk neutral valuation of all securities in the market. The possibility of risk neutral valuation leads technical traders to enter in the market. For improving the clarity of the demonstration, we assume in the following that the Brownian motion is 1-dimensional.

Definition 3 (assumption of technical traders). *Let B_t be a 1-dimensional Brownian motion with respect to $\mathcal{P}_t$, and H_s an adapted càdlàg process. The price process of the security $\mathbf{x}_t$ can be approximated with the semimartingale Z_t on $(\Omega, \mathcal{F}_t, \mathcal{P}_t)$ such that:*

$$Equil(\mathbf{x}_t) \;\approx\; Z_t \;=\; B_t + \int_0^t H_s\,ds, \tag{31}$$

with:

$$B_t \in \mathcal{M}^c_{loc}(\mathcal{P}_t) \;:=\; \{\textit{all continuous local martingales w.r.t. } \mathcal{P}_t\},$$

and:

$$\int_0^t H_s\,ds \in FV \;:=\; \{\textit{all stochastic processes with finite variation}\}.$$

Agents can evaluate the security $\mathbf{x}_t$ via the sophisticated methods of martingale theory and stochastic calculus by adopting the approximation in formula (31).

[13] See for example: Kreps [45], Harrison and Pliska [30], Heath and Jarrow [31], and Delbaen [15].

Proposition 4 (existence of an equivalent martingale measure). *Let the price process Z_t be a semimartingale of 1-dimensional Brownian motion under the measure $\mathcal{P}_t$. Then, there exists a probability measure $\mathcal{Q}_t$, equivalent to $\mathcal{P}_t$, such that Z_t can be represented as a continuous local martingale under $\mathcal{Q}_t$.*

$$\exists\, \mathcal{Q}_t \sim \mathcal{P}_t \;:\; Z_t \;=\; B_t + \int_0^t H_s\, ds \;\in\; \mathcal{M}^c_{loc}(\mathcal{Q}_t), \tag{32}$$

with $B_t \in \mathcal{M}^c_{loc}(\mathcal{P}_t)$ and $\int_0^t H_s\, ds \in FV$.

Proof. Let be $\mathcal{Q}_t = \mathcal{E}(L)_t \cdot \mathcal{P}_t$, and let $\mathcal{E}(L)_t$ be the Radon-Nikodym derivative defined as:

$$\mathcal{E}(L)_t = \exp\Big\{ L_t - \frac{1}{2}\langle L\rangle_t \Big\}, \tag{33}$$

where $\langle L\rangle_t$ is the quadratic variation of L_t and:

$$L_t = -\int_0^t H_s\, dB_s. \tag{34}$$

It results $E_{\mathcal{P}}[\mathcal{E}(L)_t] \equiv 1$. Moreover, $\mathcal{E}(L)_t$ is a martingale, because it fulfills the sufficient Novikov condition:

$$E\Big[\exp\Big\{\frac{1}{2}\langle L\rangle_t\Big\}\Big] < \infty, \qquad \forall\, t. \tag{35}$$

According to the Girsanov theorem, it results:

$$B^{\mathcal{Q}}_t = B_t - \langle B, L\rangle_t \in \mathcal{M}^c_{loc}(\mathcal{Q}_t), \tag{36}$$

where $\langle B, L\rangle_t$ is the covariation of B_t and L_t, with:

$$-\langle B, L\rangle_t = \Big\langle B, \int_0^t H_s\, dB_s \Big\rangle_t = \int_0^t H_s\, d\langle B\rangle_s = \int_0^t H_s\, ds. \tag{37}$$

Then, it results:

$$-\langle B, L\rangle_t = \int_0^t H_s\, ds \quad \Longleftrightarrow \quad B^{\mathcal{Q}}_t = Z_t \in \mathcal{M}^c_{loc}(\mathcal{Q}_t). \;\square \tag{38}$$

From the preceding proof, it follows that $(Z_t)_{t\in[0,T]}$ is square integrable:

$$\langle Z\rangle_t = \langle B\rangle_t = t < \infty \quad \Longrightarrow \quad (Z_t) \in \mathcal{L}^2. \tag{39}$$

Furthermore, according to Lévy theorem, it results that $(Z_t)_{t\in[0,T]}$ is a Brownian motion with respect to $\mathcal{Q}_t$:

$$(Z_t)_{t\in[0,T]} \in \mathcal{M}^c_{loc}, \;\; \langle Z\rangle_t = t, \;\; \forall\, t \in [0,T] \;\; \mathcal{Q}{-almost\ surely} \;\; \Rightarrow \\ \Longrightarrow \quad (Z_t)_{t\in[0,T]} \; is\ Brownian\ motion\ with\ respect\ to\ \mathcal{Q}_t. \tag{40}$$

A function is of unbounded total variation if its quadratic variation is strictly positive. Then, an immediate consequence of the Lévy theorem for Z_t is the following.

Corollary 3. *For any $t \in (0,T]$, the paths of the stochastic process Z_t are of unbounded total variation with respect to $\mathcal{Q}_t$.*

According to propositions 3 and 4, the existence of a market-clearing price at date zero leads to the existence of a risk neutral probability measure Q_t. This permits the entrance of technical traders in the market.

Definition 4 (strategy of technical traders). *We have given a security $\mathbf{x}_t$ whose price process $Equil(\mathbf{x}_t)$ can be approximated with a semimartingale Z_t. Then, risk neutral agents will be indifferent whether to buy or not the security:*

$$Equil(\mathbf{x}_t) \;\approx\; Z_t \;\Longrightarrow\; E_{\mathcal{P}_t}[Equil(\mathbf{x}_{t+dt})] \;=\; E_{\mathcal{Q}}[Z_{t+dt}] \;\Longrightarrow \\ \Longrightarrow \quad \big(\hat{r}^i_t - Equil(\mathbf{x}_t),\; \hat{\mathbf{x}}^i_t + \mathbf{x}_t\big) \;\sim_i \\ \sim_i \;\big(\hat{r}^i_t + Equil(\mathbf{x}_t),\; \hat{\mathbf{x}}^i_t - \mathbf{x}_t\big) \;\sim_i\; \big(\hat{r}^i_t,\; \hat{\mathbf{x}}^i_t\big), \\ \forall\, i \in \mathbf{TB} \cup \mathbf{TS}. \tag{41}$$

Risk neutral agents who have chosen to buy the security $\mathbf{x}_t$ *will be willing to pay a price* $\mathit{Buy}(\mathbf{x}_t) > \mathit{Equil}(\mathbf{x}_t)$*:*

$$
\begin{gathered}
\mathit{Equil}(\mathbf{x}_t) \;\approx\; Z_t \quad \Longrightarrow \\
\Rightarrow\;\; E_{\mathcal{Q}}[Z_{t+dt}] \;<\; E_{\mathcal{P}_t}[\mathit{Equil}(\mathbf{x}_{t+dt})|\,{}^{\mathit{decision\ of}}_{\mathit{buying}\,\mathbf{x}_t}] \;=\; \mathit{Buy}(\mathbf{x}_t) \;\Rightarrow \\
\Longrightarrow\;\; (\hat{r}^i_t - \mathit{Buy}(\mathbf{x}_t),\, \hat{\mathbf{x}}^i_t + \mathbf{x}_t) \;\sim_i \\
\sim_i\;\; (\hat{r}^i_t - \mathit{Equil}(\mathbf{x}_t) - \epsilon,\, \hat{\mathbf{x}}^i_t + \mathbf{x}_t\,) \;\succ_i\;\; (\hat{r}^i_t,\, \hat{\mathbf{x}}^i_t), \\
\forall\, i \in \mathbf{TB},\; \epsilon > 0.
\end{gathered} \tag{42}
$$

Risk neutral agents who have chosen to sell the security $\mathbf{x}_t$ *will be willing to accept a price* $\mathit{Sell}(\mathbf{x}_t) < \mathit{Equil}(\mathbf{x}_t)$*:*

$$
\begin{gathered}
\mathit{Equil}(\mathbf{x}_t) \;\approx\; Z_t \quad \Longrightarrow \\
\Rightarrow\;\; E_{\mathcal{Q}}[Z_{t+dt}] \;>\; E_{\mathcal{P}_t}[\mathit{Equil}(\mathbf{x}_{t+dt})|\,{}^{\mathit{decision\ of}}_{\mathit{selling}\,\mathbf{x}_t}] \;=\; \mathit{Sell}(\mathbf{x}_t) \;\Rightarrow \\
\Longrightarrow\;\; (\hat{r}^j_t + \mathit{Sell}(\mathbf{x}_t),\, \hat{\mathbf{x}}^j_t - \mathbf{x}_t) \;\sim_j \\
\sim_j\;\; (\hat{r}^j_t + \mathit{Equil}(\mathbf{x}_t) + \epsilon,\, \hat{\mathbf{x}}^j_t - \mathbf{x}_t\,) \;\succ_j\;\; (\hat{r}^j_t,\, \hat{\mathbf{x}}^j_t), \\
\forall\, j \in \mathbf{TS},\; \epsilon > 0.
\end{gathered} \tag{43}
$$

As exposed in section 3.3, the element ϵ in formulae (42) and (43) can be interpreted as the Bayesian updating of price expectations of technical agents due to their choice of buying or selling. Moreover, we can interpret ϵ as a small sum that agents should add or subtract to $Equil\,(\mathbf{x}_t)$ to make sure that their market orders would be executed.

Risk neutral investors who trades randomly according to the martingale equivalent probability measure that can be inferred from market prices, can be seen as a particular kind of noise traders which differs from its classic definition in literature[14] in the fact that they are rational and informed. A consequence of proposition 4 and corollary 3 is that the market price function is no longer unbounded if technical traders are operating. This leads to the following proposition.

Proposition 5. *In a market with fundamental and technical traders, the price of the security* $\mathbf{x}_t$ *can assume a value outside the set of Pareto-efficient*

[14] See for example Black [4] and De Long et al. [14].

bargains of fundamental traders.

$$\textit{Equil}(\mathbf{x}_t) \approx Z_t \ \wedge \ \exists\, i, j \in \mathbf{TB}_t \cup \mathbf{TS}_t \ \Rightarrow \ P[\textit{Equil}(\mathbf{x}_t) \notin \mathbf{E}_t] > 0, \tag{44}$$

where $\mathbf{TB}_t$ and $\mathbf{TS}_t$ represent, respectively, the sets of technical buyers and technical sellers at date t.

Proof. Let us suppose that the clearing-market price for a security $\mathbf{x}_t$ exists at date t and is equal to the highest reservation price among buyers that are fundamental traders, $\max[\textit{Bid}_{\mathrm{FB}}(\mathbf{x}_t)]$. All sellers and at least one buyer among fundamental traders are then operating; also, according to proposition 4, risk neutral investors can choose to act as technical traders according to the martingale equivalent probability measure that emerges from market prices.

Let us now suppose that at time t the number of technical traders for which:

$$\begin{gathered} \textit{Equil}(\tilde{\mathbf{x}}_t) \approx Z_t, \\ E_{\mathcal{P}_t}\big[\textit{Equil}(\mathbf{x}_{t+dt})\big|\,{}^{\text{decision of}}_{\text{buying } \mathbf{x}_t}\big] > E_{\mathcal{Q}}\big[Z_{t+dt}\big], \ \forall\, i \in \mathbf{TB}_t, \end{gathered} \tag{45}$$

exceeds the number of technical traders for which:

$$\begin{gathered} \textit{Equil}(\tilde{\mathbf{x}}_t) \approx Z_t, \\ E_{\mathcal{P}_t}\big[\textit{Equil}(\mathbf{x}_{t+dt})\big|\,{}^{\text{decision of}}_{\text{selling } \mathbf{x}_t}\big] < E_{\mathcal{Q}}\big[Z_{t+dt}\big], \ \forall\, j \in \mathbf{TS}_t. \end{gathered} \tag{46}$$

This means that the number of technical traders who are willing to buy the security $\mathbf{x}_t$ at the market-clearing price, exceeds at date t the number of technical traders who are willing to sell it. Thus, by assumption in section 3.4 and in definition 4, the market-clearing price raises; then it may exceed $\max[\textit{Bid}_{\mathrm{FB}}(\mathbf{x}_t)]$ and it may exit form the space of Pareto-efficient bargains $\mathbf{E}_t$. At the new price, no buyer that is fundamental trader is operating. The price will increase until technical traders who are willing to buy exceeds those who are willing to sell, and it will decrease vice versa. □

4.4 Market conditions

According to the definition of the price process in 3.5 and to proposition 5, we can now distinguish and quantitatively define the following different market conditions:

- *non speculative market*: it happens when, at a given date t, only fundamental traders are operating. The market-clearing price of a security $\mathbf{x}_t$ belongs to the set of Pareto-efficient bargains $\mathbf{E}_t$. The price function becomes:

$$f : Ask_{\mathrm{FS}}(\mathbf{x}_t) \times Bid_{\mathrm{FB}}(\mathbf{x}_t) \mapsto Equil(\mathbf{x}_t) \in \mathbf{E}_t,$$
$$\min[Ask_{\mathrm{FS}}(\mathbf{x}_t)] \leq Equil(\mathbf{x}_t) \leq \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)]. \qquad (47)$$

- *normal market*: it happens when, at a given date t, both fundamental and technical traders are operating in the market and the market-clearing price of a security $\mathbf{x}_t$ belongs to the set of Pareto-efficient bargains $\mathbf{E}_t$. The price function results:

$$f : Ask_{\mathrm{FS}}(\mathbf{x}_t) \times Bid_{\mathrm{FB}}(\mathbf{x}_t) \times Buy_{\mathrm{TB}}(\mathbf{x}_t) \times Sell_{\mathrm{TS}}(\mathbf{x}_t) \times$$
$$\times Equil(\mathbf{x}_{t-dt}) \mapsto Equil(\mathbf{x}_t) \in \mathbf{E}_t,$$
$$\min[Ask_{\mathrm{FS}}(\mathbf{x}_t)] \leq Equil(\mathbf{x}_t) \leq \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)]. \qquad (48)$$

- *market bubble*: it happens when, at a given date t, the market-clearing price of a security $\mathbf{x}_t$ exceeds $\max[Bid_{\mathrm{FB}}(\mathbf{x}_t)]$, the highest reservation price among buyers that are fundamental traders; thus, it is outside the set of Pareto-efficient bargains $\mathbf{E}_t$. Only technical traders and sellers that are fundamental traders are operating in the market. The price function becomes:

$$f : Ask_{\mathrm{FS}}(\mathbf{x}_t) \times Buy_{\mathrm{TB}}(\mathbf{x}_t) \times Sell_{\mathrm{TS}}(\mathbf{x}_t) \times$$
$$\times Equil(\mathbf{x}_{t-dt}) \mapsto Equil(\mathbf{x}_t) \notin \mathbf{E}_t,$$
$$Equil(\mathbf{x}_t) > \max[Bid_{\mathrm{FB}}(\mathbf{x}_t)]. \qquad (49)$$

- *market depression*: it happens when, at a given date t, the market-clearing price of a security $\mathbf{x}_t$ is below $\min[Ask_{\mathrm{FS}}(\mathbf{x}_t)]$, the lowest reservation price among sellers that are fundamental traders; thus, it is outside the set of Pareto-efficient bargains $\mathbf{E}_t$. Only technical traders and buyers that are fundamental traders are operating in the market. The price function becomes:

$$f: \mathit{Bid}_{\mathrm{FB}}(\mathbf{x}_t) \times \mathit{Buy}_{\mathrm{TB}}(\mathbf{x}_t) \times \mathit{Sell}_{\mathrm{TS}}(\mathbf{x}_t)\times$$
$$\times \mathit{Equil}(\mathbf{x}_{t-dt}) \mapsto \mathit{Equil}(\mathbf{x}_t) \notin \mathbf{E}_t,$$
$$\mathit{Equil}(\mathbf{x}_t) \;<\; \min[\mathit{Ask}_{\mathrm{FS}}(\mathbf{x}_t)]. \tag{50}$$

4.5 The risk neutral valuation paradox

Proposition 6. *If technical traders stop operating in a market which is either under bubble or depression conditions, the market-clearing price of the security* $\mathbf{x}_t$ *will have a jump discontinuity. The new price will assume a value inside the set of Pareto-efficient bargains of fundamental traders.*

Proof. We first prove the proposition under market bubble conditions. Let us suppose that at date t the market-clearing price of the security $\mathbf{x}_t$ is outside the space of Pareto-efficient bargains $\mathbf{E}_t$ and exceeds the highest reservation price among buyers that are fundamental traders, $\max[\mathit{Bid}_{\mathrm{FB}}(\mathbf{x}_t)]$. Thus, the market is experiencing a bubble.

Let us suppose that, while the market is still under bubble conditions, at some date $\tilde{t} > t$ all technical traders stop operating. The left and right limits of the price function in the neighborhood of $\tilde{t}$ result:

$$\lim_{t\to\tilde{t}^-} \Big[f: \mathit{Ask}_{\mathrm{FS}}(\mathbf{x}_{\tilde{t}-}) \times \mathit{Buy}_{\mathrm{TB}}(\mathbf{x}_{\tilde{t}-}) \times \mathit{Sell}_{\mathrm{TS}}(\mathbf{x}_{\tilde{t}-})\times$$
$$\times \mathit{Equil}(\mathbf{x}_{t-dt}) \mapsto \mathit{Equil}(\mathbf{x}_{\tilde{t}-})\Big],$$
$$\mathit{Equil}(\mathbf{x}_{\tilde{t}-}) \;>\; \max[\mathit{Bid}_{\mathrm{FB}}(\mathbf{x}_{\tilde{t}-})], \tag{51}$$

and:

$$\lim_{t\to\tilde{t}^+} \Big[f: \mathit{Ask}_{\mathrm{FS}}(\mathbf{x}_{\tilde{t}+}) \times \mathit{Bid}_{\mathrm{FB}}(\mathbf{x}_{\tilde{t}+}) \mapsto \mathit{Equil}(\mathbf{x}_{\tilde{t}+})\Big],$$
$$\min[\mathit{Ask}_{\mathrm{FS}}(\mathbf{x}_{\tilde{t}+})] \;\le\; \mathit{Equil}(\mathbf{x}_{\tilde{t}+}) \;\le\; \max[\mathit{Bid}_{\mathrm{FB}}(\mathbf{x}_{\tilde{t}+})]. \tag{52}$$

Then, the price function has a jump discontinuity at date $\tilde{t}$. Indeed, it results:

$$\mathit{Equil}(\mathbf{x}_{\tilde{t}-}) \;\neq\; \mathit{Equil}(\mathbf{x}_{\tilde{t}+}). \tag{53}$$

The jump consists in a negative price shock:

$$Equil(\mathbf{x}_{\tilde{t}-}) \; > \; Equil(\mathbf{x}_{\tilde{t}+}). \tag{54}$$

The proposition is then proved under market bubble conditions. It can be proved analogously under market depression. □

Please note that the market-clearing price $Equil(\mathbf{x}_t)$ will have a jump discontinuity also if all technical traders who are, alternatively, buyers or sellers will stop operating in a market which is, respectively, under bubble or depression conditions.

Theorem 1 (risk neutral valuation paradox). *Let us consider an incomplete market for derivatives, which is frictionless, informationally efficient, and has competitive rational investors with different risk attitudes. Then, if some investors are risk neutral, the market may exhibit almost surely cycles in which prices diverge from their fundamental values, ended by abrupt price adjustments towards their fundamental values.*

Proof. The demonstration of the theorem logically follows from proposition 6. □

5 A Solution to the Grossman-Stiglitz Paradox

In this section we intend to propose a solution to the paradox of Grossman and Stiglitz on the absence of a competitive equilibrium if additional costly information is available.[15] We intend to demonstrate that, within our framework, their assertion is not true. Our first step is to demonstrate that the risk neutral valuation paradox is still valid even under asymmetric information.

5.1 The theorem under asymmetric information

Proposition 7. *The arbitrage-free valuation paradox still holds in the presence of asymmetric information.*

[15] See Grossman and Stiglitz in [26], page 395, conjecture 6: in an informationally efficient market with costly information "prices convey all information, and there is no incentive to purchase information. Hence, the only possible equilibrium is one with no information. But if everyone is uninformed, it clearly pays some individual to become informed. Thus, there does not exists a competitive equilibrium."

Proof. Let X_t be a stochastic process on the measurable space $(\Omega, \mathcal{F}_t)$ representing the instant return produced by an asset $\mathbf{x}_t$ at date $t \in [0, T]$. Its realization $x_t(\omega)$ represents the contingent consumption gained or lost at date t by the owner of the asset $\mathbf{x}_t$ if the state is $\omega \in \Omega$. It results:

$$\mathbf{x}_t = X_t + \mathbf{x}_{t+dt} \tag{55}$$

Let us now assume that:

$$X_t = \theta_t + \epsilon_t, \tag{56}$$

where θ_t is an observable random variable, and ϵ_t is an unobservable random variable. Then, we can interpret θ_t as a measurement of X_t with error ϵ_t.[16] We assume that all agents have access to public information $\mathcal{F}_t$, and that some of them have also access to private information in θ_t.

Let us denote with $\mathbf{FS}_{inf}$ the set of sellers, among fundamental traders, who can observe θ_t, and with $\mathbf{FS}^c_{inf}$ its complement in $\mathbf{FS}$. Then, the set of reservation prices of sellers that are fundamental traders becomes:

$$\begin{aligned} Ask_{FS}(\mathbf{x}_t) \;=\; \Big\{ & Ask_i(\mathbf{x}_t \,|\, \mathcal{F}_t + \theta_t),\; Ask_j(\mathbf{x}_t \,|\, \mathcal{F}_t) \;\Big| \\ & \Big|\; \forall\, i \in \mathbf{FS}_{inf},\;\; \forall\, j \in \mathbf{FS}^c_{inf} \Big\}, \end{aligned} \tag{57}$$

where the reservation price $Ask_i(\mathbf{x}_t \,|\, \mathcal{F}_t + \theta_t)$ is based both on public and private information, and the reservation price $Ask_j(\mathbf{x}_t \,|\, \mathcal{F}_t)$ is based only on public information.

We denote with $\mathbf{FB}_{inf}$ the set of buyers, among fundamental traders, that can observe θ_t, and with $\mathbf{FB}^c_{inf}$ its complement in $\mathbf{FB}$. Then, the set of reservation prices of buyers that are fundamental

[16] More formally, both X_t and θ_t are measurable functions that map the measurable space $(\Omega, \mathcal{F}_t)$ into the set of real numbers $\mathbb{R}$, and whose mappings differ by ϵ_t.

traders becomes:

$$\begin{aligned} Bid_{FB}(\mathbf{x}_t) \;=\; \Big\{ & Bid_i(\mathbf{x}_t \,|\, \mathcal{F}_t + \theta_t),\; Bid_j(\mathbf{x}_t \,|\, \mathcal{F}_t) \;\Big| \\ & \Big|\; \forall\, i \in \mathbf{FB}_{inf},\;\; \forall\, j \in \mathbf{FB}^c_{inf} \Big\}, \end{aligned} \tag{58}$$

where the reservation price $Bid_i(\mathbf{x}_t \,|\, \mathcal{F}_t + \theta_t)$ is based both on public and private information, and the reservation price $Bid_j(\mathbf{x}_t \,|\, \mathcal{F}_t)$ is based only on public information.

We denote with $\mathbf{TS}_{inf}$ the set of sellers, among technical traders, who can observe θ_t, and with $\mathbf{TS}^c_{inf}$ its complement in $\mathbf{TS}$. Then, the set of market orders of sellers that are technical traders becomes:

$$\begin{aligned} Sell_{TS}(\mathbf{x}_t) = \Big\{ & E_{i\,\mathcal{P}_t}\big[Equil(\mathbf{x}_{t+dt}) \,|\, \mathcal{F}_t + \theta_t \,\wedge\, \text{selling } \mathbf{x}_t\big], \\ E_{j\,\mathcal{P}_t}\big[Equil(\mathbf{x}_{t+dt}) \,|\, \mathcal{F}_t \,\wedge\, \text{selling } \mathbf{x}_t\big] & \,\Big|\, \forall\, i \in \mathbf{TS}_{inf}, \forall\, j \in \mathbf{TS}^c_{inf} \Big\}, \end{aligned} \tag{59}$$

where the future price expectations $E_{i\,\mathcal{P}_t}[\,\cdot\,]$ and $E_{j\,\mathcal{P}_t}[\,\cdot\,]$ are based both on the choice of selling the security (by definition 4) and, respectively, on public and private information and on public information only.

We denote with $\mathbf{TB}_{inf}$ the set of buyers, among the technical traders, who can observe θ_t, and with $\mathbf{TB}^c_{inf}$ its complement in $\mathbf{TB}$. Then, the set of market orders of buyers that are technical traders becomes:

$$\begin{aligned} Buy_{TB}(\mathbf{x}_t) = \Big\{ & E_{i\,\mathcal{P}_t}\big[Equil(\mathbf{x}_{t+dt}) \,|\, \mathcal{F}_t + \theta_t \,\wedge\, \text{buying } \mathbf{x}_t\big], \\ E_{j\,\mathcal{P}_t}\big[Equil(\mathbf{x}_{t+dt}) \,|\, \mathcal{F}_t \wedge \text{buying } \mathbf{x}_t\big] & \,\Big|\, \forall\, i \in \mathbf{TB}_{inf}, \forall\, j \in \mathbf{TB}^c_{inf} \Big\}, \end{aligned} \tag{60}$$

where the future price expectations $E_{i\,\mathcal{P}_t}[\,\cdot\,]$ and $E_{j\,\mathcal{P}_t}[\,\cdot\,]$ are based both on the choice of buying the security (by definition 4) and, respectively, on public and private information and on public information only.

The strategies of individuals that have access to both public and private information are conceptually identical to those of individuals who have only access to public information. Then, the demonstration of the risk neutral valuation paradox under asymmetric information logically follows from the main demonstration by substituting the new definitions of $Ask_S(\mathbf{x}_t)$, $Bid_B(\mathbf{x}_t)$ and $E_M[Z_{t+dt}]$. $\square$

5.2 Conditions for buying additional costly information

In the following, we expose the conditions for buying the private information θ_t for the different categories of traders.

A seller $i \in \mathbf{FS}$ that is fundamental trader is willing to buy the information θ_t at a cost c if it results:

$$U_i\left(\hat{r}_t^i + Ask_i(\mathbf{x}_t \mid \mathcal{F}_t + \theta_t) - c,\ \hat{\mathbf{x}}_t^i - \theta_t - \epsilon_t - \mathbf{x}_{t+dt}\right) \geq$$
$$\geq U_i\left(\hat{r}_t^i + Ask_i(\mathbf{x}_t \mid \mathcal{F}_t),\ \hat{\mathbf{x}}_t^i - X_t - \mathbf{x}_{t+dt}\right). \quad (61)$$

A buyer $i \in \mathbf{FB}$ that is fundamental trader is willing to buy the information θ_t at a cost c if it results:

$$U_i\left(\hat{r}_t^i - Bid_i(\mathbf{x}_t \mid \mathcal{F}_t + \theta_t) - c,\ \hat{\mathbf{x}}_t^i + \theta_t + \epsilon_t + \mathbf{x}_{t+dt}\right) \geq$$
$$\geq U_i\left(\hat{r}_t^i - Bid_i(\mathbf{x}_t \mid \mathcal{F}_t),\ \hat{\mathbf{x}}_t^i + X_t + \mathbf{x}_{t+dt}\right). \quad (62)$$

A buyer $i \in \mathbf{TB}$ that is tchnical trader is willing to buy the information θ_t at a cost c if it results:

$$U_i\left(\hat{r}_t^i - Buy_i(\mathbf{x}_t \mid \mathcal{F}_t + \theta_t) - c,\ \hat{\mathbf{x}}_t^i + \theta_t + \epsilon_t + \mathbf{x}_{t+dt}\right) \geq$$
$$\geq U_i\left(\hat{r}_t^i - Buy_i(\mathbf{x}_t \mid \mathcal{F}_t),\ \hat{\mathbf{x}}_t^i + X_t + \mathbf{x}_{t+dt}\right). \quad (63)$$

This condition is never fulfilled when the additional information is costly. Indeed, given that the price process *Equil*$(\mathbf{x}_t)$ can be approximated with a semimartingale Z_t, it results:

$$Equil(\mathbf{x}_t) \approx Z_t \Rightarrow E_{\mathcal{Q}}[\,\hat{\mathbf{x}}_t^i + \theta_t + \epsilon_t + \mathbf{x}_{t+dt}\,] = E_{\mathcal{Q}}[\,\hat{\mathbf{x}}_t^i + X_t + \mathbf{x}_{t+dt}\,], \tag{64}$$

and, according to definition 4:

$$Equil(\mathbf{x}_t) \approx Z_t \Rightarrow Buy_i(\mathbf{x}_t \,|\, \mathcal{F}_t + \theta_t) = Buy_i(\mathbf{x}_t \,|\, \mathcal{F}_t). \tag{65}$$

Then, the condition in formula (63) is satisfied only for $c = 0$. Thus, no buyer and (analogously) no seller among technical traders will be willing to buy additional costly information but they will prefer to take advantage of the informative content of prices.

In the preceding equations, the stochastic term X_t is substituted with the sum $\theta_t + \epsilon_t$ in change of a decrease of c in certain current consumption. Please note that, with the payment of c, the term θ_t becomes observable and turns from stochastic to deterministic. Then, we can interpret the cost of information c as the insurance premium paid by risk-averse agents for reducing the stochastic component from X_t to ϵ_t.

5.3 A solution to the Grossman-Stiglitz paradox

The Grossman-Stiglitz paradox states that, in an informationally efficient market, all relevant information is already reflected in market prices, thus no single agent would have sufficient incentive to acquire the information on which prices are based. For example, if this statement were correct, in a complete derivatives market there would be no incentive to forecast volatility, because future volatility would already be included in market prices of options.

According to Grossman and Stiglitz [26], the conjecture would lead to the absence of a competitive equilibrium if additional costly information is available.We intend to demonstrate that, within our framework, their assertion is not true.

Proposition 8. *In a market for the security $\mathbf{x}_t$, with fundamental and technical traders, the availability of additional costly information does not prevent the market from reaching a competitive equilibrium.*

Proof. We start from the demonstration of the proposed theorem with asymmetric information, in sect. 5.1. Let X_t be a stochastic process on the measurable space $(\Omega, \mathcal{F}_t)$ representing the instant return produced by an asset $\mathbf{x}_t$ at date $t \in [0, T]$, with:

$$\mathbf{x}_t = X_t + \mathbf{x}_{t+dt} = \theta_t + \epsilon_t + \mathbf{x}_{t+dt} \tag{66}$$

where θ_t and ϵ_t are respectively an observable and an unobservable random variable. We can interpret θ_t as a measurement of X_t with error ϵ_t. Let us suppose that the observation of θ_t is only possible at a cost c, while the public information $\mathcal{F}_t$ is freely available.

Technical traders take advantage of the informative content of prices and are not interested in buying additional costly information. On the other hand, at each time t, fundamental traders choose whether buying or not the additional information; their choices depend on the cost of information and their subjective risk preferences, expressed by their subjective expected utility function U_i. Then, their strategies change into the following.

Let us denote with $\mathbf{FS}_{inf}$ the set of sellers, among fundamental traders, that have paid to observe θ_t, and with $\mathbf{FS}^c_{inf}$ its complement in $\mathbf{FS}$. Their strategy is to submit limited orders of selling at no less than their subjective reservation price, $Ask(\mathbf{x}_t)$, determined as:

$$U_i\left(\hat{r}^i_t + Ask_i(\mathbf{x}_t|\mathcal{F}_t + \theta_t),\, \hat{\mathbf{x}}^i_t - \mathbf{x}_t\right) \;=\; U_i(\hat{r}^i_t,\, \hat{\mathbf{x}}^i_t), \;\; \forall\, i \in \mathbf{FS}_{inf}, \tag{67}$$

and

$$U_j\left(\hat{r}^j_t + Ask_j(\mathbf{x}_t|\mathcal{F}_t),\, \hat{\mathbf{x}}^j_t - \mathbf{x}_t\right) \;=\; U_j(\hat{r}^j_t,\, \hat{\mathbf{x}}^j_t), \;\; \forall\, j \in \mathbf{FS}^c_{inf}, \tag{68}$$

where the limited order $Ask_i(\mathbf{x}_t \,|\, \mathcal{F}_t + \theta_t)$ is based on both public information $\mathcal{F}_t$ and private information θ_t, and $Ask_j(\mathbf{x}_t \,|\, \mathcal{F}_t)$ is based on public information only.

We denote with $\mathbf{FB}_{inf}$ the set of buyers, among fundamental traders, that have paid to observe θ_t, and with $\mathbf{FB}^c_{inf}$ its complement in $\mathbf{FB}$. Their strategy is to submit limited orders of buying at no more than their subjective reservation price, $Bid(\mathbf{x}_t)$, determined as:

$$U_i\left(\hat{r}_t^i - Bid_i(\mathbf{x}_t|\mathcal{F}_t + \theta_t),\, \hat{\mathbf{x}}_t^i + \mathbf{x}_t\right) = U_i(\hat{r}_t^i,\, \hat{\mathbf{x}}_t^i), \quad \forall\, i \in \mathbf{FB}_{inf}, \tag{69}$$

and

$$U_j\left(\hat{r}_t^j - Bid_j(\mathbf{x}_t|\mathcal{F}_t),\, \hat{\mathbf{x}}_t^j + \mathbf{x}_t\right) = U_j(\hat{r}_t^j,\, \hat{\mathbf{x}}_t^j), \quad \forall\, j \in \mathbf{FB}_{inf}^c, \tag{70}$$

where the limited order $Bid_i(\mathbf{x}_t \mid \mathcal{F}_t + \theta_t)$ is based on both public and private information, and $Bid_j(\mathbf{x}_t \mid \mathcal{F}_t)$ is based on public information only.

Given the above strategies of fundamental traders, we have a market with asymmetric information as described in section 5.1. Then, the demonstration of the risk neutral valuation paradox, when additional costly information is available, logically follows from proposition 7 and from the main demonstration in section 4.5. □

The solution to the Grossman-Stiglitz paradox comes from the fact that its main premise is only partially true. According to the risk neutral valuation paradox, in an informationally efficient market all relevant information *is not* already reflected in market prices. Indeed, the market-clearing price fails to convey information from individuals whose orders are not executed. Clear examples within our framework are the market conditions of bubble and depression. Under market bubble conditions, the price function is independent from orders coming from buyers that are fundamental traders; see also formula (51). Similarly, under market depression conditions, the price function is independent from orders coming from sellers that are fundamental traders.

6 Conclusions

In this paper we have highlighted the role of risk neutral investors in generating endogenous bubbles in derivatives markets. If an incomplete market for derivatives has no arbitrage opportunities, it is indifferent for risk neutral investors to trade or not and, in the former case, to buy or to sell any particular security. Hence, their trading choices will vary randomly over time. Prices can then diverge from their fundamental values but, whenever risk neutral investors stop trading (which can happen almost surely in a sufficiently long period of time), prices will have abrupt adjustments towards their fundamental values. Our main result consists in an original theorem of mathematical finance that we have called the risk neutral valuation paradox.

The theorem is a consequence of the fundamental theorem of asset pricing and the related martingale theory of bubbles. It shows that derivatives markets, which are informationally efficient, incomplete and with some risk neutral investors, may exhibit violations to the Gaussian random walk hypothesis. Furthermore, it suggests that extreme price movements like price peaks or crashes may happen with a higher than-normal frequency.

We have demonstrated the theorem in the alternative cases of symmetric and asymmetric information. Then, we have proposed a solution to the Grossman-Stiglitz paradox on the value of information. We have observed that the cost of private information can be seen as an insurance premium that agents may pay for reducing the stochastic component of their investments, according to their subjective risk preferences.

The theorem yields surprisingly concrete implications for the theory of financial and derivatives markets. We suggest four possible advances. The first is that violations to the random walk hypothesis can be compatible with informationally efficient markets. The second is that the informative content of prices in almost perfect markets may decrease when risk neutral investors are operating. The third is that risk neutral investors, even when they are rational and informed, may stimulate the emergence of boom-and-bust cycles. The fourth is that, under some circumstances, the distribution of price movements in an almost perfect market may differ from a normal distribution.

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Finito di stampare nel mese di Settembre 2011
Presso **studiografico&stampadigitale Copy Right**
Via Torre Tonda 8 – Tel. 079.200395 – Fax 079.4360444
07100 Sassari

www.crenos.it

ISBN 978-88-84-67-674-0

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